# ON THE HELLY PROPERTY OF SOME INTERSECTION GRAPHS

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SOBRE A PROPRIEDADE HELLY DE ALGUNS GRAFOS DE INTERSEÇÃO

Tanilson Dias dos Santos

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Orientadores: Jayme Luiz Szwarcfiter
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Programa: Engenharia de Sistemas e Computação

Um grafo EPG é um grafo de aresta-interseção de caminhos sobre uma grade. Nesta tese de doutorado exploraremos principalmente os grafos EPG, em particular os grafos $B_1$-EPG. Entretanto, outras classes de grafos de interseção serão estudadas, como as classes de grafos VPG, EPT e VPT, além dos parâmetros número de Helly e número de Helly forte nos grafos EPG e VPG. Apresentaremos uma prova de $NP$-completude para o problema de reconhecimento de grafos $B_1$-EPG-Helly. Investigamos os parâmetros número de Helly e o número de Helly forte nessas duas classes de grafos, EPG e VPG, a fim de determinar limites inferiores e superiores para esses parâmetros. Resolvemos completamente o problema de determinar o número de Helly e o número de Helly forte para os grafos $B_k$-EPG e $B_k$-VPG, para cada valor $k$.

Em seguida, apresentamos o resultado de que todo grafo $B_1$-EPG Chordal está simultaneamente nas classes de grafos VPT e EPT. Em particular, descrevemos estruturas que ocorrem em grafos $B_1$-EPG que não suportam uma representação $B_1$-EPG-Helly e assim definimos alguns conjuntos de subgrafos que delimitam subfamílias Helly. Além disso, também são apresentadas características de algumas famílias de grafos não triviais que estão propriamente contidas em $B_1$-EPG-Helly.

Palavras-chave: EPG, EPT, Grafos de Interseção, $NP$-completude, Propriedade Helly, VPG, VPT.

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ON THE HELLY PROPERTY OF SOME INTERSECTION GRAPHS

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An EPG graph $G$ is an edge-intersection graph of paths on a grid. In this doctoral thesis we will mainly explore the EPG graphs, in particular $B_1$-EPG graphs. However, other classes of intersection graphs will be studied such as VPG, EPT and VPT graph classes, in addition to the parameters Helly number and strong Helly number to EPG and VPG graphs. We will present the proof of $NP$-completeness to Helly-$B_1$-EPG graph recognition problem. We investigate the parameters Helly number and the strong Helly number in both graph classes, EPG and VPG in order to determine lower bounds and upper bounds for this parameters. We completely solve the problem of determining the Helly and strong Helly numbers, for $B_k$-EPG, and $B_k$-VPG graphs, for each value $k$.

Next, we present the result that every Chordal $B_1$-EPG graph is simultaneously in the VPT and EPT graph classes. In particular, we describe structures that occur in $B_1$-EPG graphs that do not support a Helly-$B_1$-EPG representation and thus we define some sets of subgraphs that delimit Helly subfamilies. In addition, features of some non-trivial graph families that are properly contained in Helly-$B_1$ EPG are also presented.

Keywords: EPG, EPT, Helly property, Intersection graphs, $NP$-completeness, VPG, VPT.

# Contents

# Chapter 1

# Introduction

*Believe and you will understand;*
*faith precedes, follows intelligence.*

---

Saint Augustine

Graph Theory is a branch of Mathematics that is used by the Computer Science to describe and model several real and theoretical problems. This doctoral thesis is dedicated to solving some problems of Graph Theory. In particular, in this chapter, you will find a brief description of the related problems, the motivation of the study and, a summary of the organization of the text.

*Graph Theory* is based on relations between points that we call vertices interconnected (by elements denoted as edges) in a network. In this context we define a graph $G = (V, E)$, where $V(G)$ denotes the vertex set of $G$ and $E(G)$ its edge set. The graph is the object that we use to model the relationship among elements of a set.

An *intersection graph* is a graph that represents the pattern of intersections of a family of sets. A graph $G$ can be represented as an intersection graph when for each vertex $v_i, v_j$ of $G$ there are corresponding sets $S_i, S_j$ such that $S_i \cap S_j \neq \emptyset$ if and only if $(v_i, v_j) \in E(G)$. In this doctoral thesis, we are interested in the study of intersection graphs. Issues related to intersection graphs have been attracting the attention of researchers since the 1960, e.g. [33], and to the present day, see [62, 64].

First, we know that every graph is an intersection graph, i.e. any graph can be represented by some intersection model, [33, 72]. SCHEINERMAN [68] presents research that is exclusively dedicated to the characterization of classes of intersection graphs, also providing necessary and sufficient conditions for the existence of intersection representations for some specific graph classes.

Many important graph families can be described as intersection graphs. We can cite Interval, Circular-arc, Permutation, Trapezoid, Chordal, Disk, Circle graphs which are among the most important or at least the most studied classes in the literature in general.

Interval graphs are the intersection graph class of a collection of segments on a line, and the class of Chordal graphs corresponds to the graphs where each cycle $C_n, n \geq 3$ has a chord. Interval graphs have been extensively studied by [54]. About Chordal graphs, GAVRIL [38] shows that this class corresponds exactly to the intersection graph of subtrees on a tree. In this thesis, we will study intersection graphs of paths on a grid and on trees.

GOLUMBIC *et al.* [46] defined the edge intersection graphs of paths on a grid (EPG graphs). Similarly, [7, 9] defined the vertex intersection graphs of paths on a grid (VPG graphs). Both intersection models have some practical importance since they can be used to generalize naturally the context of circuit layout problems and layout optimization [69] where a layout is modeled as paths (wires) on a grid. Thus, they are problems that arise directly from this modeling: reducing the number of times that each path can bend in order to minimize the cost or difficulty of production of a microchip or electronic board [10, 58]; or other times layouts may consist of several layers where the paths on each layer are not allowed to intersect, this can be understood as a coloring problem. These are the main applications that instigate research on the EPG and VPG representations of some graph families. Other applications and details on circuit layout problems can be found in [10, 58, 69].

Some particular questions related to intersection graphs aroused our research interest. Among these, we can mention: "What is the complexity of recognizing a class of path intersection graphs on a grid if we restrict the number of bends in each path individually and considering the fact of each set of intersections have a common element?"; "Will it be possible to solve the problem of calculating some parameters in the class of paths intersection graphs on a grid even when the entire paths bend $k$ times?"; "Is there any relationship among the classes of intersection graphs when we change the tree host to a grid host?". The answers to these and other questions are considered in the next chapters of this thesis.

The text of this thesis is distributed over the next 5 chapters as follows.

Chapter 2 contains the definitions and concepts needed to fully understand this work. In addition, we provide a short overview of the problems studied and a brief literature review on the main subjects covered in the text.

Chapter 3 will be dedicated to the study of the Helly property and EPG graphs. In particular, the chapter presents an analysis of some basic EPG representations, a comparison of $L$-shaped paths and $B_1$-EPG graph classes, as well as a proof of the $NP$-completeness of the Helly-$B_1$-EPG graph recognition problem [17].

In Chapter 4, the parameters Helly number and strong Helly number will be studied for $B_k$-EPG and $B_k$-VPG graphs. We used the strategy of determining tight lower and upper bounds to show the value of the Helly and strong Helly number parameters in each class and for each value of $k$.

Chapter 5 presents relationship among Chordal $B_1$-EPG, VPT and EPT graphs. We show that if a graph $G$ is a $B_1$-EPG graph that is $\{S_3, S_3', S_3'', C_4\}$-free then $G$ is Helly-$B_1$ EPG. We also show some non-trivial graph classes that are Helly-$B_1$ EPG, namely Bipartite, Blocks, Cactus, and Line of Bipartite. The main result of this chapter is proof that every Chordal $B_1$-EPG graph is simultaneously in the VPT and EPT classes. The manuscript of this chapter and corresponding research was done while the author of this doctoral thesis was a doctoral research fellow at the National University of La Plata - UNLP, Math Department.

Chapter 5 contains other paper that has been submitted to the journal Discussiones Mathematicae Graph Theory (DMGT).

Chapter 6 is dedicated to discussing the results of this research and it includes the concluding remarks of this thesis with suggestions for future work.

The following are the manuscripts produced during this thesis:

1. BORNSTEIN, C. F.; GOLUMBIC, M.C.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. The Complexity of Helly-B1-EPG graph Recognition. In: Discrete Mathematics & Theoretical Computer Science (DMTCS), Source: oai:arXiv.org:1906.11185, June 4, 2020, vol. 22 no. 1.

2. BORNSTEIN, C. F.; MORGENSTERN, G.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. Helly and Strong Helly Numbers of $B_k$-EPG and $B_k$-VPG Graphs. To be submitted to a journal.

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2. BORNSTEIN, C. F.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. Sobre a Dificuldade de Reconhecimento de Grafos B1-EPG-Helly. In: XXXVIII Congresso da Sociedade Brasileira de Computação, 2018, Natal - RN. Computação e Sustentabilidade, 2018. p. 113-116.

3. BORNSTEIN, C. F.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. The complexity of B1-EPG-Helly graph recognition. In: VIII Latin American Workshop On Cliques in Graphs (LAWCG), ICM 2018 Satellite Event, 2018, Rio de Janeiro. Program and Abstracts, 2018. p. 69.

4. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. Identifying Subclasses of Helly-$B_1$-EPG Graphs. 52nd Brazilian Operational Research Symposium (SBPO), 2020.

5. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. On Subclasses of Helly-$B_1$-EPG Graphs. Reunión Anual de la Unión Matemática Argentina (virtUMA), 2020.

6. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. Paths on Hosts: $B_1$-EPG, EPT and VPT Graphs. Submitted to: Latin American Workshop on Cliques in Graphs (LAWCG), 2020.

The results obtained in our research can be found in the set of manuscripts previously cited and in this doctoral thesis. For each one of the Chapters 3, 4 and 5 there is a brief introduction and a related paper.

Next, we present the basic concepts.

# Chapter 2

# Intersection graphs of paths on grid and trees

*If you know the enemy and know yourself, you need not fear the result of a hundred battles. If you know yourself but not the enemy, for every victory gained you will also suffer a defeat. If you know neither the enemy nor yourself, you will succumb in every battle.*

---

Sun Tzu, The Art of War

> In this chapter, we will present some concepts that will facilitate the understanding of the studied problems. In particular, we describe the notations and we will illustrate with examples only those concepts and definitions that are outside the basic scope of graph theory. As a basic bibliography on graphs, algorithms, and NP-completeness we suggest reading [14] and [73] .

In this thesis, we will consider finite graphs, connected and simple, i.e. graphs without loops (edge connecting a vertex in itself) or more than one edge connecting two vertices. Thus, when we talk about graphs we will consider a simple, finite and connected graph unless something different is explicitly said.

Next, we describe the terminology and notation used in this work.

A *graph* $G$ is a structure composed of two finite sets: $V(G)$ is a non-empty set whose elements are called *vertices*, and $E(G)$ is a set of unordered pairs of distinct elements taken from $V(G)$, which are called *edges*. An edge $e = (u, v) \in E(G)$ is formed by the pair of vertices $u, v \in V(G)$, in this case $u$ and $v$ are said to be *adjacent* vertices . We also say that $e$ is an *incident edge* to $u$ and $v$. We denote the *cardinality* of $|V(G)| = n$ and $|E(G)| = m$.

Given a vertex $v \in V(G)$, $N(v)$ and $N[v]$ represent the *open* and the *closed neighborhood* of $v$ in $G$, respectively. For a subset $S \subseteq V(G)$, $G[S]$ is the subgraph of $G$ induced by $S$. If $\mathcal{F}$ is any family of graphs, we say that $G$ is *$\mathcal{F}$-free* if $G$ has no induced subgraph isomorphic to a member of $\mathcal{F}$.

Let $u, v$ be vertices of $G$, if $N(u) = N(v)$ then $u$ and $v$ are said to be *false twins*, on the other hand, if $N[u] = N[v]$, then $u$ and $v$ are said *true twins*. The *degree of a vertex* $v$ is denoted by $d(v)$ and corresponds to the number of vertices adjacent to $v$, i.e., the cardinality of $|N(v)|$. The *maximum degree* of a graph $G$ is denoted by $\Delta(G) = \max\{d(v) \mid v \in V(G)\}$. Similarly, the *minimum degree* is denoted by $\delta(G) = \min\{d(v) \mid v \in V(G)\}$.

Given a graph $G$, and a vertex $v \in V(G)$, the graph $G \backslash \{v\}$ is obtained from $G$ by removing the vertex $v$ from its vertex set, and also removing all edges of $E(G)$ incident to $v$. Similarly, given an edge $e \in E(G)$, the graph $G \backslash \{e\}$ is obtained from $G$ removing the edge $e$ from $E(G)$.

We say that $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ when $V' \subseteq V$ and $E' \subseteq E$. When the subgraph $G'$ contains all edges of $E$ whose ends are contained in $V'$, then $G'$ is the *induced subgraph* of $G$ by $V'$.

A graph $G$ is a *cycle*, denoted by $C_n$, if it is a sequence of vertices $v_1, \dots, v_n, v_1$, where $v_i \neq v_j$ for $i \neq j$ and $(v_i, v_{i+1}) \in E(G)$, such that $n \geq 3$. For a cycle $C_k$, we say that it is an *even cycle* if $k$ is even and an *odd cycle*, otherwise. We say that an edge $e_{ij}$ is *between* two vertices $v_i$ and $v_j$ when $e_{ij}$ is incident edge to $v_i$ and $v_j$. A *chord* is an edge that is between two non-consecutive vertices in the sequence of vertices of a cycle. An *induced cycle* or *chordless cycle* is a cycle that has no chord. A graph that has no cycles is called *acyclic*. A graph $G$ is *connected* if there is a path between any pair of vertices of $G$. A graph is a *tree* when it is acyclic and connected. A connected subgraph of a tree is called *subtree*.

*Chordal graphs* are the graphs where each induced cycle $C_n, n \geq 3$ has a chord.

A graph $G$ formed by an induced cycle $H$ plus a single universal vertex $v$ connected to all vertices of $H$ is called *wheel graph*. If the wheel has $n$ vertices, it is denoted by $n$-wheel.

A *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non adjacent vertices.

The *k-sun graph* $S_k$, $k \geq 3$, consists of $2k$ vertices, an independent set $X = \{x_1, \dots, x_k\}$ and a clique $Y = \{y_1, \dots, y_k\}$, and edges set $E_1 \cup E_2$, where $E_1 = \{(x_1, y_1); (y_1, x_2); (x_2, y_2); (y_2, x_3); \dots, (x_k, y_k); (y_k, x_1)\}$ forms the outer cycle and $E_2 = \{(y_i, y_j) | i \neq j\}$ forms the inner clique.

A set $\mathcal{S}$ is *maximal* in relation to a particular property $P$ if $\mathcal{S}$ satisfies $P$, and each set $S'$ containing properly $\mathcal{S}$ does not satisfy $P$. In a similar way, a set $\mathcal{S}$ is *minimal* in relation to a particular property $P$ if $\mathcal{S}$ satisfies $P$, and each subset $S'$

that is properly contained in $\mathcal{S}$ does not satisfy $P$.

A graph $G$ is an *intersection graph* of a family of subsets of a set $\mathcal{S}$, when it is possible to associate each vertex $v \in V(G)$ to a subset $S_v \subseteq \mathcal{S}$, such that $S_u \cap S_v \neq \emptyset$ if and only if $(u, v) \in E(G)$. In this thesis, in particular, we will study four families of intersection graphs: the VPG, EPG, VPT and EPT graphs.

The term *grid* is used to denote the Euclidean space of integers orthogonal coordinates. Each pair of integers *coordinates* corresponds to a point or *vertex of the grid* (which by the context is not to be confused with the vertex of the graph). The term *grid edge* (which is also not to be confused with the edge of the graph), will be used to denote a pair of vertices that are at distance one in the grid. Two edges $e_1$ and $e_2$ are *consecutive edges* when they share exactly one point on the grid. A grid is the *host* on which we accommodate the VPG and EPG representations. When we refer to the VPT and EPT graphs, we implicitly know that the host of their representations is a tree.

A *path in the grid* is distinguished by two contexts, in the first we study families of subsets $\mathcal{F}$ of edge of the grid. In this context a path in the grid is defined as a finite sequence of consecutive edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), \ldots, e_i = (v_i, v_{i+1}), \ldots, e_m = (v_m, v_{m+1})$, where $v_i \neq v_j$ for $i \neq j$. We call a collection of such paths an *EPG representation*, i.e., a collection of paths that represent a graph via its intersection graph (considering edge intersections). *EPG graphs* are the class of graphs that admit an EPG representation. In the second context, for vertex paths, we study families of subsets $\mathcal{F}$ of vertex of the grid, and a path consists of a sequence of consecutive vertices of the grid $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1})$ is an edge of the grid, for all $i \in 1, \ldots, k-1$, where $v_i \neq v_j$ for $i \neq j$, and a collection of these paths forms a *VPG representation* and corresponds to a *VPG graph*.

The first and last edges of a path are called *extremity edges*. The *direction of an edge* is *vertical* when the first coordinate of its vertices is equal, and is *horizontal* when the second coordinate is equal. A *bend* in a path is a pair of consecutive edges $e_1, e_2$ of the path, such that the directions of $e_1$ and $e_2$ are different. When two edges $e_1$ and $e_2$ form a bend, they are called *bend edges*. A *segment* is a path without bend.

In the context of EPG graphs, we say that two paths are *edge-intersecting*, or simply *intersecting*, if these share at least one edge (of the grid).

EPG graphs are a class of intersection graphs of paths on a grid [46]. Shortly after came the VPG graphs, this class was introduced in 2011 [9] and [7]. These classes consist of graphs whose vertices can be represented by paths of a grid $Q$, such that two vertices of $G$ are adjacent if and only if the corresponding paths intersect (in edges, if EPG graphs or in vertex, if VPG graphs). If every path in a representation can be represented with a maximum of $k$ bends, we say that this graph $G$ has a $B_k$-*EPG* (resp. $B_k$-*VPG*) representation. When $k = 1$ we say that this is a *single bend* representation.

Let $P$ be a family of paths on a host tree $T$. Two types of intersection graphs from the pair $< P, T >$ are defined, namely VPT and EPT graphs. The *edge intersection graph* of $P$, EPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in EPT(P) if and only if the corresponding paths in $P$ share at least one edge in T. Similarly, the *vertex intersection graph* of $P$, VPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in VPT(P) if and only if the corresponding paths in $P$ share at least one vertex in $T$. VPT and EPT graphs are incomparable families of graphs. However, when the maximum degree of the host tree is restricted to three the family of VPT graphs coincides with the family of EPT graphs [41]. Also, it is known that any Chordal EPT graph is VPT (see [71]). Recall that it was shown that Chordal graphs are the vertex intersection graphs of subtrees of a tree [38].

Let $\mathcal{F}$ be a family of subsets of some universal set $U$, and $h$ an integer $\geq 1$. Say that $\mathcal{F}$ is *h-intersecting* when every group of $h$ sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$ is the intersection of all sets of $\mathcal{F}$, denoted $core(\mathcal{F})$.

The family $\mathcal{F}$ is *h-Helly* when every $h$-intersecting subfamily $\mathcal{F}'$ of it satisfies $core(\mathcal{F}') \neq \emptyset$, see e.g. [31]. On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are $h$ subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong h-Helly*. Clearly, if $\mathcal{F}$ is $h$-Helly then it is $h'$-Helly, for $h' \geq h$. Similarly, if $\mathcal{F}$ is strong $h$-Helly then it is strong $h'$-Helly, for $h' \geq h$.

Finally, the *Helly number* of the family $\mathcal{F}$ is the least integer $h$, such that $\mathcal{F}$ is $h$-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least $h$, for which $\mathcal{F}$ is strong $h$-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number.

A *class* $\mathcal{C}$ of families $\mathcal{F}$ of subsets of some universal set $U$ is a subcollection of the families $\mathcal{F}$ of $U$. Say that $\mathcal{C}$ is a *hereditary* class when it closed under inclusion. The *Helly number* of a class $\mathcal{C}$ of families $\mathcal{F}$ of subsets is the largest Helly number among the families $\mathcal{F}$. Similarly, the *strong Helly number* of a class $\mathcal{C}$ is the largest strong Helly number of the families of $\mathcal{C}$.

If $\mathcal{F}$ is a family of subsets and $\mathcal{C}$ a class of families, denote by $H(\mathcal{F})$ and $H(\mathcal{C})$, the Helly numbers of $\mathcal{F}$ and $\mathcal{C}$, respectively, while $sH(\mathcal{F})$ and $sH(\mathcal{C})$ represent the strong Helly numbers of $\mathcal{F}$ and $\mathcal{C}$.

We say that a family of sets is *pairwise intersecting*, i.e. two by two intersecting if any two sets in the family intersect. A collection $C$ of non-empty sets satisfies the Helly property, i.e. it is 2-Helly, when every subcollection pairwise intersecting $S$ of $C$ has at least one element that is in every subset of $S$.

For simplicity of notation, in this thesis when we refer to a family of sets as a

Helly family it is understood that this family is 2-Helly.

We say that a path $P_i$ is a *$B_k$-path* if it contains at most $k$ bends. Say that $\mathcal{F}$ is a $B_k$-paths family, or simply a $B_k$-family, if each path of $\mathcal{F}$ is a $B_k$-path.

In Boolean algebra, a *clause* is a disjunction or conjunction of literals. We say that a *formula* $F$ is in the *Conjunctive Normal Form* (CNF) if $F$ is a conjunction of clauses, where a clause is a disjunction of literals.

## 2.1 Related Works

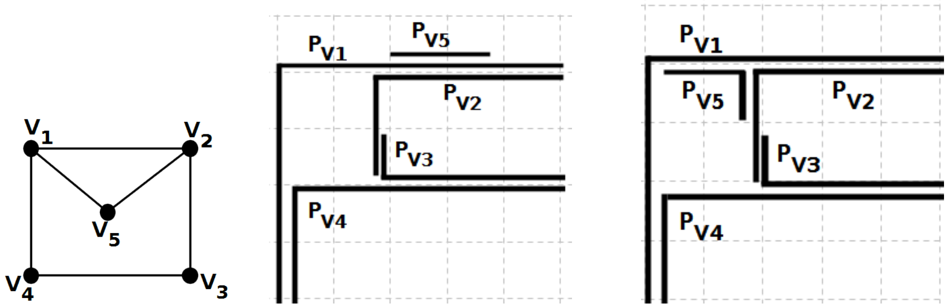
In this section, we will present the main known results on the related study topics in this work, namely Helly property, EPG, VPG, EPT, and VPT graphs.

### 2.1.1 On the Helly property

The Helly property is named in honor of the Austrian mathematician Eduard Helly, who in 1923 proposed a famous theorem about the relationship of intersecting sets. Such a theorem motivates the so-called *Helly property* which can be stated as follows: given a collection of sets $C$, not empty, we say that this collection satisfies the Helly property when every subcollection of $C$ that is pairwise intersecting has at least one element in common.

We can note that the Helly property is a topic that has instigated scientific research since it appeared, moreover, we can also mention recent works in the area of Graph Theory, see [11, 12, 28, 30, 47, 51, 64]. The study of the Helly property proves to be useful in the most diverse areas of science, of which one can enumerate applications in semantics, code theory, computational biology, database, image processing, graph theory, optimization, in problems of location and linear programming, [51]. In particular, in the area of Graph Theory, the Helly property has motivated the study of several graph classes, for example, we can cite the Clique-Helly graphs [30], Helly Circular-arc [66], Helly EPT [6], Disk-Helly [56] and Helly Hypergraphs [60].

In addition to the applications mentioned above, the Helly property can be studied on $B_k$-EPG representations, where each path is considered as a set of edges. A graph $G$ has a Helly-$B_k$-EPG representation if there is a $B_k$-EPG representation of $G$ where each path has at most $k$ bends and the representation satisfies the Helly property. We will use the notation $P_{v_i}$ to indicate the path corresponding to the vertex $v_i$. Figure 2.1(a) depicts two representations $B_1$-EPG of a graph with 5 vertices. Figure 2.1(b) depicts pairwise intersecting paths $(P_{v_1}, P_{v_2}, P_{v_5})$, containing a common edge, so this is a Helly-$B_1$-EPG representation. In Figure 2.1(c), although the 3 paths are pairwise intersecting, there is no edge common to the 3 paths simultaneously, and thus they do not satisfy the Helly property.




(a) A graph with 5 vertices. (b) A $B_1$-EPG representation that satisfies the Helly property. (c) A $B_1$-EPG representation that does not satisfy the Helly property.

Figure 2.1: A graph with 5 vertices in (a) and some single bend representations: Helly in (b) and not Helly in (c).

In this thesis, we are interested in EPG representations of graphs that satisfy the Helly property. In particular, for the $B_1$-EPG graphs, this directly implies that each clique has a special format, and the paths that compose it always share an edge of the representation in the grid, i.e. an edge-clique. Using this premise we were able to present Helly subfamilies for $B_1$-EPG graphs and we also presented a hardness proof in recognizing this class of graphs. We will also study within the scope of this research the parameters Helly number and strong Helly number in paths on a grid.

### 2.1.2 On EPG graphs

A problem related to the study of EPG graphs is the problem of edge-intersection graphs of paths in a tree, well known in the literature as EPT (Edge-intersection Graphs of Paths in a Tree), see for instance [38, 43]. For EPT graphs, in particular, the value of the parameters Helly number, which is 2, and the strong Helly number, which is 3, are known results, also in [43]. The parameters Helly number and strong Helly number had been studied in EPT graphs when the set of paths satisfies the Helly property, see [62] and [63].

Regarding the complexity of the $B_k$-EPG graph recognition, only the hardness recognition of a few of these graph subclasses was determined. $B_0$-EPG can be recognized in polynomial time, since these correspond to the interval graphs, see [16]. In contrast, the $B_1$-EPG and $B_2$-EPG graphs recognition are $NP$-complete problems, see [48, 61], and the $B_1$-EPG graph recognition problem remains $NP$-complete even for $L$-shaped paths on a grid, see [21]. Moreover, in this doctoral thesis you will also find an $NP$-completeness proof for the Helly-$B_1$-EPG graphs recognition in Chapter 3, and the same chapter we further studied the subsets of $L$-shapes and its relationship with $B_1$-EPG and Helly-$B_1$-EPG graphs.

In this work, we are going to study graphs that have a Helly-EPG representation and related subjects. The Helly property related to EPG graph representations was studied by [46] and [47]. In particular, they determined the parameter strong Helly number of graphs $B_1$-EPG. We determine two parameters to every class of EPG graphs, the Helly number and strong Helly number, these results are presents in Chapter 4.

The *bend number* of a graph $G$ is the smallest $k$ for which $G$ is a $B_k$-EPG graph. Analogously, the bend number of a class of graphs is the smallest $k$ for which all graphs in the class have a $B_k$-EPG representation. Interval graphs have bend number 0, trees have bend number 1, see [46], and outerplanar graphs have bend number 2, see [49]. The bend number for the class of planar graphs is still open, but according to [49], it is either 3 or 4.

Research about graphs of edge-intersection of paths on a grid is a relatively new topic in the area of Graph Theory. The first formal definitions of problems and applications were presented by Golumbic in 2009 [46]. Since then, several branches of researches have been conducted by the scientific community. These questions often discuss the path representations, restrictions on the bend number in a representation, among others. A survey that summarizes the state-of-the-art for the topic of EPG graphs can be found at [25].

Next, we present some results regarding the *bend number* for some classes of graphs, among others.

In their study, ALCÓN *et al.* [4], the authors show that 3 bends are enough to represent all graphs in the class of circular-arc graphs, i.e. they are in $B_3$-EPG. Additionally, they also show that there are circular-arc graphs that cannot be represented with 2 bends. Using the fact that we can to represent any circular-arc graph using only a rectangle of a grid of any size, the work defines the class of EPR graphs and classifies the normal circular-arc graphs as being $B_2$-EPR, they also show that there are normal circular-arc graphs that are not $B_1$-EPR. Finally, the work gives a characterization of $B_1$-EPR graphs by a minimal family of forbidden induced subgraphs and shows that this subfamily corresponds to a subclass of normal Helly Circular-arc graphs.

In the paper of BIEDL and STERN [13], the authors show that 5 bends are enough to represent all planar graphs and that 3 bends are enough to represent all outerplanar graphs. These results are further improved by [49]. In addition to these results, the work shows that every Bipartite Planar graph has a $B_2$-EPG representation and that every Line graph has a $B_2$-EPG representation. In this thesis, we demonstrate that every Line of Bipartite graph is in Helly-$B_1$ EPG, these results are in Chapter 5.

HELDT *et al.* in [49] showed that 4 bends are enough to represent all planar graphs and present a linear algorithm to find this representation with 4 bends.

However, the authors still comment that for some planar graphs, 3 bends are often enough to construct the representation. In fact, it is not that simple the majority of planar graphs could be constructed with 4 bends, in fact, there are no known planar graphs that cannot be drawn using 3 bends. This leaves the question: if 4 bends are always enough to represent any planar graph, then are 4 bends really needed to represent any planar graph? That question is still open. The authors still conjecture that there is a graph where for any of its EPG representations there is always at least one path that needs to use the 4 bends.

The Table 2.1 presents the main known bounds for the *bend number*, denoted by $b(G)$, of some graph classes.

Table 2.1: Some graph classes and known bounds to their *bend number*.

| Graph Class | b(G) | Reference |
|---|---|---|
| Interval graphs | 0 | [46] |
| Forests, Cycles | 1 | [47] |
| Outerplanar | 2 | [49] |
| Planar | $\in [3, 4]$ | [49] |
| Bipartite Planar | 2 | [13] |
| Line Graph | 2 | [13] |
| dgn(G) [1]$\leq k$ | $2k - 1$ | [49] |
| tw(G) [2]$\leq k$ | $2k - 2$ | [49] |
| Degree $\leq \Delta$ | $\in [\lceil \frac{\Delta}{2} \rceil, \Delta]$ | [49] |
| Circular-arc | 3 | [4] |
| Normal Circular-arc | 2 | [4] |
| Halin graphs | 2 | [35] |

In addition to the results cited for bounds on the bend number of some classes of graphs, there are many works that characterize other types of graphs not mentioned in this table, such that the work of RIES in [65] that characterizes the Chordal graphs claw-free, bull-free and diamond-fee that have a $B_1$-EPG representation. In that same article, there is also a characterization of some Split graphs, with a restriction on the size of the independent set or clique, by forbidden subgraphs. The work still has an interesting result that shows that the neighborhood of every vertex of a graph $B_1$-EPG induces a graph that is Weakly Chordal. Implicitly this paper delimits a set of Helly-$B_1$-EPG graphs, the bull-free graphs. Based on this fact in this thesis, we extend the results to delimit another Helly-$B_1$-EPG subfamily, the diamond-free subfamily. This result can be found in Chapter 5.

Although it is possible to find several lines of researches on EPG graphs inves-

[1] Degeneracy
[2] Treewidth

tigating the bend number, the interests of studies in this class of graphs extend to other classic problems, which we can mention to follow.

In COHEN *et al.* [26] a linear time recognition algorithm is presented for $B_1$-EPG Cographs. The paper characterize $B_1$-EPG Cographs and $B_0$-VPG Cographs by a family of forbidden induced subgraphs. The algorithm that the paper presents uses the Cotree of the Cograph in the recognition process.

Approximation Algorithms for coloring $B_1$-EPG graphs were studied in [32]. The work cited shows that the coloring problem and the maximum independent set problem are both $NP$-complete for graphs $B_1$-EPG even when the EPG representation is given. The authors present a 4-approximate algorithm that solves both problems, assuming that the EPG representation is given. The work still shows that the maximum clique can be found efficiently in graphs $B_1$-EPG even when the representation is not given.

Clique coloring problems in $B_1$-EPG graphs were studied by [15]. The authors consider the clique coloring problem and show that $B_1$-EPG graphs are 4-clique-colorables and present a linear time algorithm to solve the problem. Moreover, given a $B_1$-EPG representation of a graph, the paper provides a linear time algorithm that constructs a 4-clique coloring of it.

We can also mention as an often research with respect to EPG graphs the study of $NP$-hardness [49, 61], area of the grid necessary to represent a graph whose maximum degree is $\Delta(G)$ [8], and many others. The hardness of recognizing few classes of EPG graphs is known, and even for small $k$ values only. Research with EPG graphs whose representations satisfy the Helly property is sparse. Thus, these topics and other similar topics prove to be interesting branches of research from a scientific point of view.

Finally, we mention that the $B_k$-EPG hierarchy is proper, i.e.,

$B_0$-EPG $\subset$ $B_1$-EPG $\subset$ $B_2$-EPG $\subset \ldots$ $B_{k-1}$-EPG $\subset$ $B_k$-EPG $\subset$ $B_{k+1}$-EPG

this result is demonstrated by BIEDL and STERN [13] for even $k$ and HELDT *et al.* [48] complete the result for all $k$. A correlated result is presented by ASINOWSKI and SUK [8] that proved that for any $k$, only a small fraction of all labeled graphs on $n$ vertices are $B_k$-EPG.

### 2.1.3 On VPG graphs

VPG representations arise naturally when studying circuit layout problems and layout optimization where layouts are modeled as paths (wires) on grids. One approach to minimize the cost or difficulty of production involves minimizing the number of times that each path bend, see [10, 58, 69]. Other times layout may consist of several layers where the paths on each layer are not allowed to intersect. This is naturally modeled as the coloring problem on the corresponding intersection graph, see [5].

A graph is a VPG if it is the vertex intersection graph of paths in a grid. A graph is called $B_k$-VPG if it has a $B_k$-VPG representation, i.e. if there is a representation where each path in this representation has at most $k$ bends. VPG graphs were introduced in 2011 by ASINOWSKI *et al.* [9] and ASINOWSKI *et al.* [7]. They prove that VPG and String are the same graph class. However, it is known that recognizing String graphs is an $NP$-complete problem, by the results of [52, 67].

ASINOWSKI *et al.* [7] study $B_0$-VPG graphs and observe that horizontal and vertical segments have strong Helly number 2 and that the clique problem has polynomial-time complexity, given the path representation. Among other results, they present proof that the recognition and coloring problems for $B_0$-VPG graphs are $NP$-complete. Moreover, they give a 2-approximation algorithm for coloring $B_0$-VPG graphs. Furthermore, they prove that triangle-free $B_0$-VPG graphs are 4-colorable, and this is the best possible. In addition, they present a hierarchy of VPG graphs relating them to other known families of graphs, see Figure 2.2. The grid intersection graphs are shown to be equivalent to the bipartite $B_0$-VPG graphs and the circle graphs are strictly contained in $B_1$-VPG. They still prove the strict containment of $B_0$-VPG into $B_1$-VPG, and conjecture that, in general, this strict containment continues for all values of $k$. Finally, they present a graph that is not in $B_1$-VPG.

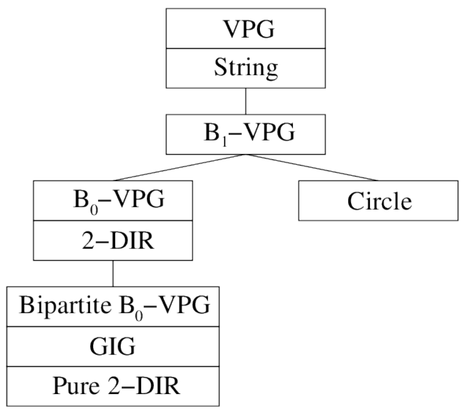



Figure 2.2: Relations between $B_k$-VPG graphs and well known graph classes [7].

It is known that all planar graphs are $B_2$-VPG, see [22]. This paper also shows that the 4-connected planar graphs constitute a subclass of the intersection graphs of $Z$-shapes (i.e., a special case of $B_2$-VPG). Additionally, they demonstrate that a $B_2$-VPG representation of a planar graph can be constructed in polynomial time. They further show that the triangle-free planar graphs are contact graphs of L-shapes, Γ-shapes, vertical segments, and horizontal segments (i.e., a special case of contact $B_1$-VPG).

Approximation algorithms for the maximum independent set problem over the class of $B_1$-VPG graphs are presented by LAHIRI *et al.* [53]. Also, the NP-completeness of the decision version restricted to unit length equilateral $B_1$-VPG graphs was established by them.

COHEN *et al.* [27] investigate the VPG graphs, and specifically the relationship between the bend number of a Cocomparability graph and the poset dimension of its complement. They show that the bend number of a Cocomparability graph $G$ is at most the poset dimension of the complement of $G$ minus one. Then, via Ramsey type arguments, they show that their upper bound is best possible.

In FELSNER *et al.* [34], the authors research the L-shapes representations for $B_k$-VPG graphs. The paper investigates several known subclasses of segment graphs (SEG-graphs), motivated mainly by research [57] that states that every $[\llcorner, \ulcorner]$-shape is an SEG-graph. They show that these subclasses of SEG-graphs belong to $[\llcorner]$-shapes, also that all Planar 3-trees, all Line graphs of Planar graphs, and all full subdivisions of Planar graphs are $[\llcorner]$-shapes. Furthermore, FELSNER *et al.* [34] showed that the complement of Planar graphs is $B_{17}$-VPG graphs and complements of full subdivisions of the latter class are $B_2$-VPG graphs.

In the paper of GOLUMBIC and RIES [42] certain subclasses of $B_0$-VPG graphs have been characterized and showed to admit polynomial-time recognition. We can list these classes as Split, Chordal claw-free, and Chordal bull-free $B_0$-VPG graphs. The $B_0$-VPG Split graphs were characterized by a set of forbidden induced subgraphs.

In CHAPLICK *et al.* [23], they investigate $B_0$-VPG graphs. Their paper describes a polynomial time algorithms for recognizing Chordal $B_0$-VPG graphs, and for recognizing $B_0$-VPG graphs that have representation on a grid with 2 rows and an arbitrary number of columns.

CHAPLICK *et al.* [24] show that for every fixed $k$, $B_k$-VPG $\subsetneq$ $B_{k+1}$-VPG and that recognition of graphs from $B_k$-VPG is $NP$-complete even when the input graph is given by a $B_{k+1}$-VPG representation.

$B_0$-VPG graphs restricted to Block graphs were studied by ALCÓN *et al.* [5]. Their research has given a characterization by an infinite family of minimal forbidden induced subgraph for $B_0$-VPG Block graphs. Furthermore, the work provides an alternative recognition and representation algorithm for $B_0$-VPG graphs also in the class of Block graphs.

In Chapter 4 we study the parameters Helly number and strong Helly number for $B_k$-VPG graphs. We determine the value of these parameters to $k = 0, 1, 2, 3$ and verify that they are unbounded for $k \geq 4$.

### 2.1.4 On EPT and VPT graphs

Models based on paths intersection may consider intersections by vertices or intersections by edges. Cases where the paths are hosted on a tree appear in [36, 39, 41]. Representations using paths on a grid were considered in [42, 46, 47].

EPT and VPT graphs have applications in communication networks, see [18] in [20]. Assume that we model a communication network as a tree $T$ and the message routes to be delivered in this communication network as paths on $T$. Two paths conflict if they both require to use the same link (vertex). This conflict model is equivalent to an EPT (a VPT) graph. Suppose we try to find a schedule for the messages such that no two messages sharing a link (vertex) are scheduled in the same time interval. Then a vertex coloring of the EPT (VPT) graph corresponds to a feasible schedule on this network, [18] and [20].

Let $P$ be a family of paths on a host tree $T$. Two types of intersection graphs from the pair $<P, T>$ are defined, namely VPT and EPT graphs. The *edge intersection graph* of $P$, EPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in EPT(P) if and only if the corresponding paths in $P$ share at least one edge in T. Similarly, the *vertex intersection graph* of $P$, VPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in VPT(P) if and only if the corresponding paths in $P$ share at least one vertex in $T$.

VPT and EPT graphs are incomparable families of graphs. However, when the maximum degree of the host tree is restricted to three the family of VPT graphs coincide with the family of EPT graphs [41]. Also, it is known that any Chordal EPT graph is VPT, see [71]. Recall that it was shown that Chordal graphs are the vertex intersection graphs of subtrees of a tree [38].

Next, we list some research involving EPT and VPT graphs.

Although VPT graphs can be characterized by a fixed number of forbidden subgraphs, see [55], it is shown that EPT graphs recognition is an NP-complete problem, see [39]. Main optimization and decision problems such as recognition [36], the maximum clique [37] , the minimum vertex coloring [40] and the maximum stable set problems [70] are polynomial-time solvable in VPT whereas recognition and minimum vertex coloring problems remain NP-complete in EPT graphs [41]. In contrast, we can solve in polynomial time the maximum clique, see [39], and the maximum stable set, see [74], problems in EPT graphs.

In ALCÓN *et al.* [1] we find a short paper that deals with EPT graphs. The paper defines the concept of satellite of a clique and we give us a necessary condition for the structure of cliques in EPT graphs based on satellites of cliques. In addition, the paper presents a finite family of minimal forbidden subgraphs for the EPT class.

Next, we will present the notation $[h, s, t]$ so that we can talk about some equivalences known in the literature.

The class of graphs that have an $[h, s, t]$-representation is denoted by $[h, s, t]$. A graph $G$ has an $[h, s, t]$-representation when $h, s$, and $t$ are positive integers such that $h \geq s$, there is a host tree $T$ with maximum degree $\Delta(T) \leq h$, there is a family of subtrees $S = \{S_u \subseteq T / u \in V(G)\}$ with $\Delta(S_u) \leq s$, and there is an edge $uv \in E(G)$ if and only if $|S_u \cap S_v| \geq t$.

In ALCÓN *et al.* [3] was studied a set of minimal forbidden induced subgraphs from VPT and their $[h, s, t]$-representations. When there is no restriction on the maximum degree of $T$ or on the maximum degree of the subtrees is used the notation $h = \infty$ and $s = \infty$, respectively. Therefore, $[\infty, \infty, 1]$ is the class of Chordal graphs and $[2, 2, 1]$ is the class of interval graphs. The classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$ correspond to VPT and EPT respectively in [41]; and UV and UE, respectively in [59]. By taking $h = 3$ they obtain a characterization by minimal forbidden induced subgraphs of the class VPT $\cap$ EPT = EPT $\cap$ Chordal = $[3, 2, 2] = [3, 2, 1]$, see GOLUMBIC and JAMISON [41]. The paper also proved that the problem of deciding whether a given VPT graph belongs to $[h, 2, 1]$ is NP-complete even when restricted to the class VPT $\cap$ Split without dominated stable vertices, among other minor results.

Priscila Petito in her master thesis [62] researched UE graphs, UV graphs, and the Helly property. In particular, when it considers the UE family with Helly property its study leads to a new graph class denoted by UEH graph class. The master thesis also presents results to directed and rooted trees. Furthermore, the master thesis also considers the relationship among these classes in addition to others. In time, the work still considers the parameter strong Helly number in its scope. This doctoral thesis approaches a similar branch of research since we studied EPG and Helly graphs, the parameters Helly number and Strong Helly number, and also VPT and EPT (UV and UE respectively) graphs.

In Priscila Petito's doctoral thesis [63] UEH graphs were studied. The work presents a characterization by forbidden subgraphs that are simultaneously UEH and Split. Among the main problems addressed in the research are also the clique coloring problem in UEH graphs, the study of the complexity of the sandwich problem for the Clique-Helly class. In addition, the work also studies the inclusion relations among UE, UEH, and Clique-Helly classes.

Returning to $[h, s, t]$ notation, it is known that when the EPT graphs are restricted to host trees of vertex degree 3 this class corresponds precisely to the Chordal EPT graphs. In GOLUMBIC *et al.* [45] was proved an analogous result that Weakly Chordal EPT graphs are precisely the EPT graphs whose host tree restricted to degree 4. Moreover, they provide an algorithm to reduce a given EPT representation of a Weakly Chordal EPT graph to an EPT representation on a degree 4 tree. In short, their proof state that $[4, 2, 2]$ graphs are equivalent to Weakly Chordal $[\infty, 2, 2]$ graphs. In addition, we know that when the maximum degree of the host tree T is 3, the coloring problem is polynomial, by [39]. The paper of [45] also shows the analogous polynomial result for a degree 4 host tree, thus the coloring problem on EPT graphs restricted to a host tree of vertex degree 4 is polynomial.

In GOLUMBIC *et al.* [44], the research presents equivalences and the complete hierarchy of intersection graphs of paths in a tree, this including VPT and EPT grpahs, in particular orthodox-$[h, s, t]$ graphs with $s = 2$ and considering variations of $h, t$. For more information about orthodox-$[h, s, t]$ graphs we recommend reading JAMISON and MULDER [50] and PINTO [64].

Other researches still focus on variations of the EPT representations, such as [18] and [19]. These two articles represent the same research divided into two parts. Given a set of paths $P$, they define the graph ENPT($P$) of edge intersecting non-splitting paths of a tree, denoted by ENPT graph, as the graph having a vertex for each path in $P$, and an edge between every pair of vertices representing two paths that are both edge-intersecting and non-splitting. A graph $G$ is an ENPT graph if there is a tree $T$ and a set of paths $P$ of $T$ such that $G$ = ENPT($P$). The papers investigate the basic properties of this class and proof that some graph classes belong to ENPT, such that Trees, Holes, Complete graphs, etc. Among the results, they show that the problem of finding such a representation is $NP$-Hard in general also for this class.

As we can see, EPT and VPT graphs have been extensively studied in the literature. With approaches that study from classic problems in these classes of graphs to variations of constructions and representations in those same classes. In this thesis, in particular, we will study the relationship of the VPT and EPT graphs with the EPG graphs.

In Chapter 5 we consider relationship between classes VPT, EPT and Chordal $B_1$-EPG graphs.

In the next section, we present a table with the main notations used in the text.

## 2.2 Terminology

Table 2.2 describes the basic symbols and their meanings about graph theory. More specific definitions will be given in the next chapters as necessary.

In the following chapters, we will dedicate ourselves to expose the main results obtained by researching this thesis.

Table 2.2: Terms and basic symbols of Graph Theory used in this thesis.

| Symbol | Description |
|---|---|
| $G = (V, E)$ | Graph $G$ with vertex set $V(G)$ and edge set $E(G)$. |
| $V(G)$ | Vertex set of $G$. |
| $E(G)$ | Edge set of $G$. |
| $n(G)$ | Number of vertices in $G$. |
| $m(G)$ | Number of edges in $G$. |
| $v_i$ | Vertex $v_i$. |
| $P_{v_i}$ | Path corresponding to the vertex $v_i$. |
| $e = (v_i, v_j)$ | Edge $e$ with endpoints $v_i$ and $v_j$. |
| $d(v)$ | Degree of vertex $v$. |
| $\delta(G)$ | Minimum degree of a vertex in $G$. |
| $\Delta(G)$ | Maximum degree of a vertex in $G$. |
| $N(v)$ | Opened neighborhood of the vertex $v$. |
| $N[v]$ | Closed neighborhood of the vertex $v$. |
| $G[S]$ | Induced subgraph in $G$ by subset of vertices $S$. |
| $\|S\|$ | Cardinality of set $S$. |
| $G \backslash \{v\}$ | Subgraph obtained of $G$ by removing the vertex $v$. |
| $C_n$ | Induced Cycle with $n$ vertices. |
| $W_n$ | Wheel graph with $n$ vertices. |
| $K_{r,s}$ | Complete Bipartite graph with parts os size $r$ and $s$. |
| $K_n$ | Complete graph or clique with $n$ vertices. |
| $B_k$-representation | Representation where each path has at most $k$ bends. |
| $< P, T >$ | Set of paths $P$ on a tree $T$. |
| $[h, s, t]$-representation | Representation on a host tree of degree at most $h$ of subtrees of degree at most $s$ and intersection of lenght at least $t$. |

# Chapter 3

# The Helly property and EPG graphs

*Genius is one percent inspiration, ninety nine percent perspiration.*

Thomas Edison

In this chapter, after a brief introduction, the reader can find a complete version of the paper published in the journal DMTCS. We will examine the hierarchical relationships among some EPG and Helly-EPG classes. Besides, we will approach $B_1$-EPG representations of some graphs that will be used later. First, let us focus our attention to understand how the classes $B_0$-EPG, $B_1$-EPG, Helly-$B_1$ EPG, and $L$-shaped paths are related, then we consider the $B_1$-EPG representations of graphs $C_4$ and the Octahedral graph. Finally, we will present the proof of $NP$-completeness for the Helly-$B_1$-EPG graph recognition problem.

## 3.1 Introduction

An EPG graph $G$ is a graph that admits a representation in which its vertices are represented by paths of a grid $Q$, such that two vertices of $G$ are adjacent if and only if the corresponding paths have at least one common edge.

The study of EPG graphs has motivation related to the problem of VLSI design that combines the notion of edge intersection graphs of paths in a tree with a VLSI grid layout model, see [46]. The number of bends in an integrated circuit may increase the layout area, and consequently, increase the cost of chip manufacturing. This is one of the main applications that instigate research on the EPG representations of some graph families when there are constraints on the number of bends in the paths used in the representation. Other applications and details on circuit layout problems can be found in [10, 58].

In this chapter, we study the Helly-$B_k$-EPG graphs. First, we show that every graph admits an EPG representation that is Helly, and present a characterization of Helly-$B_1$-EPG representations. Besides, we relate Helly-$B_1$-EPG graphs with L-shaped graphs, a natural family of subclasses of $B_1$-EPG. Finally, we prove that recognizing Helly-$B_k$-EPG graphs is in NP, for every fixed $k$. Besides, we show that recognizing Helly-$B_1$-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs. Other results found in the chapter are as follows: we show that every graph admits a Helly-EPG representation, and $\frac{\mu}{2n} - 1 \leq b_H(G) \leq \mu - 1$; and that Helly-$B_k$-EPG $\subsetneq$ $B_k$-EPG for each $k > 0$.

Next, we present the main paper that gave rise to this chapter.

## 3.2 Article published in the Discrete Mathematics & Theoretical Computer Science (DMTCS) journal.

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# *The Complexity of Helly-$B_1$-EPG graph Recognition*[*]

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Golumbic, Lipshteyn, and Stern defined in 2009 the class of EPG graphs, the intersection graph class of edge paths on a grid. An EPG graph $G$ is a graph that admits a representation where its vertices correspond to paths in a grid $Q$, such that two vertices of $G$ are adjacent if and only if their corresponding paths in $Q$ have a common edge. If the paths in the representation have at most $k$ bends, we say that it is a $B_k$-EPG representation. A collection $C$ of sets satisfies the Helly property when every sub-collection of $C$ that is pairwise intersecting has at least one common element. In this paper, we show that given a graph $G$ and an integer $k$, the problem of determining whether $G$ admits a $B_k$-EPG representation whose edge-intersections of paths satisfy the Helly property, so-called Helly-$B_k$-EPG representation, is in NP, for every $k$ bounded by a polynomial function of $|V(G)|$. Moreover, we show that the problem of recognizing Helly-$B_1$-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

**Keywords:** paths, grid, EPG, Helly, intersection graphs, NP-completeness, single bend.

## 1 Introduction

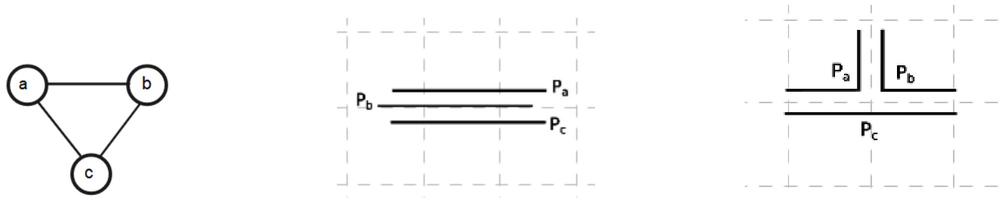
An EPG graph $G$ is a graph that admits a representation in which its vertices are represented by paths of a grid $Q$, such that two vertices of $G$ are adjacent if and only if the corresponding paths have at least one common edge.

The study of EPG graphs has motivation related to the problem of VLSI design that combines the notion of edge intersection graphs of paths in a tree with a VLSI grid layout model, see Golumbic et al. (2009). The number of bends in an integrated circuit may increase the layout area, and consequently,

[*]This work is partially supported by Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - Brasil (FAPERJ) - grant E-26/203.272/2017; Conselho Nacional de Desenvolvimento Científico e Tecnológico – Brasil (CNPq) - grant 303726/2017-2; and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) - Finance Code 001.

increase the cost of chip manufacturing. This is one of the main applications that instigate research on the EPG representations of some graph families when there are constraints on the number of bends in the paths used in the representation. Other applications and details on circuit layout problems can be found in Bandy and Sarrafzadeh (1990); Molitor (1991).

A graph is a $B_k$-EPG graph if it admits a representation in which each path has at most $k$ bends. As an example, Figure 1(a) shows a $C_3$, Figure 1(b) shows an EPG representation where the paths have no bends and Figure 1(c) shows a representation with at most one bend per path. Consequently, $C_3$ is a $B_0$-EPG graph. More generally, $B_0$-EPG graphs coincide with interval graphs.




(a) The graph $C_3$ (b) $B_0$-EPG representation of $C_3$ (edge-clique) (c) $B_1$-EPG representation of $C_3$ (claw-clique)

**Fig. 1:** The graph $C_3$ and representations without bends and with 1 bend

The *bend number* of a graph $G$ is the smallest $k$ for which $G$ is a $B_k$-EPG graph. Analogously, the bend number of a class of graphs is the smallest $k$ for which all graphs in the class have a $B_k$-EPG representation. Interval graphs have bend number $0$, trees have bend number $1$, see Golumbic et al. (2009), and outerplanar graphs have bend number $2$, see Heldt et al. (2014a). The bend number for the class of planar graphs is still open, but according to Heldt et al. (2014a), it is either $3$ or $4$.

The class of EPG graphs has been studied in several papers, such as Alcón et al. (2016); Asinowski and Suk (2009); Cohen et al. (2014); Golumbic et al. (2009); Heldt et al. (2014b); Pergel and Rzążewski (2017); Golumbic and Morgenstern (2019), among others. The investigations regarding EPG graphs frequently approach characterizations concerning the number of bends of the graph representations. Regarding the complexity of recognizing $B_k$-EPG graphs, only the complexity of recognizing a few of these sub-classes of EPG graphs have been determined: $B_0$-EPG graphs can be recognized in polynomial time, since it corresponds to the class of interval graphs, see Booth and Lueker (1976); in contrast, recognizing $B_1$-EPG and $B_2$-EPG graphs are NP-complete problems, see Heldt et al. (2014b) and Pergel and Rzążewski (2017), respectively. Also, note that the paths in a $B_1$-EPG representation have one of the following shapes: └, ┘, ┌ and ┐. Cameron et al. (2016) showed that for each $S \subset \{$└, ┘, ┌, ┐$\}$, it is NP-complete to determine if a given graph $G$ has a $B_1$-EPG representation using only paths with shape in $S$.

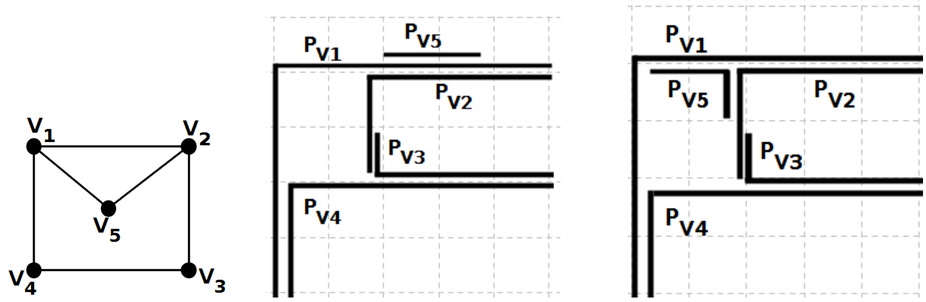
A collection $C$ of sets satisfies the Helly property when every sub-collection of $C$ that is pairwise intersecting has at least one common element. The study of the Helly property is useful in diverse areas of science. We can enumerate applications in semantics, code theory, computational biology, database, image processing, graph theory, optimization, and linear programming, see Dourado et al. (2009).

The Helly property can also be applied to the $B_k$-EPG representation problem, where each path is considered a set of edges. A graph $G$ has a Helly-$B_k$-EPG representation if there is a $B_k$-EPG representation of $G$ where each path has at most $k$ bends, and this representation satisfies the Helly property. Figure 2(a) presents two $B_1$-EPG representations of a graph with five vertices. Figure 2(b) illustrates 3 pairwise intersecting paths $(P_{v_1}, P_{v_2}, P_{v_5})$, containing a common edge, so it is a Helly-$B_1$-EPG representation. In Figure 2(c), although the three paths are pairwise intersecting, there is no common edge in all three paths, and therefore they do not satisfy the Helly property.

The Helly property related to EPG representations of graphs has been studied in Golumbic et al. (2009) and Golumbic et al. (2013).

Let $\mathcal{F}$ be a family of subsets of some universal set $U$, and $h \geq 2$ be an integer. Say that $\mathcal{F}$ is $h$-*intersecting* when every group of $h$ sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$, denoted by $core(\mathcal{F})$, is the intersection of all sets of $\mathcal{F}$. The family $\mathcal{F}$ is $h$-*Helly* when every $h$-intersecting subfamily $\mathcal{F}'$ of $\mathcal{F}$ satisfies $core(\mathcal{F}') \neq \emptyset$, see e.g. Duchet (1976). On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are $h$ subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong* $h$-*Helly*. Note that the Helly property that we will consider in this paper is precisely the property of being 2-Helly.

The *Helly number* of the family $\mathcal{F}$ is the least integer $h$, such that $\mathcal{F}$ is $h$-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least $h$, for which $\mathcal{F}$ is strong $h$-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number. In Golumbic et al. (2009) and Golumbic et al. (2013), they have determined the strong Helly number of $B_1$-EPG graphs.




(a) A graph with 5 vertices

(b) $B_1$-EPG representation that satisfies the Helly property

(c) $B_1$-EPG representation that does not satisfy the Helly property

**Fig. 2:** A graph with 5 vertices in (a) and some single bend representations: Helly in (b) and not Helly in (c)

Next, we describe some terminology and notation.

The term *grid* is used to denote the Euclidean space of integer orthogonal coordinates. Each pair of integer coordinates corresponds to a *point* (or vertex) of the grid. The *size* of a grid is its number of points. The term *edge of the grid* will be used to denote a pair of vertices that are at a distance one in the grid. Two edges $e_1$ and $e_2$ are *consecutive edges* when they share exactly one point of the grid. A (simple) path in the grid is as a sequence of distinct edges $e_1, e_2, \leq, e_m$, where consecutive edges are adjacent, i.e., contain a common vertex, whereas non-consecutive edges are not adjacent. In this context, two paths only intersect if they have at least a common edge. The first and last edges of a path are called *extremity edges*.

The *direction of an edge* is vertical when the first coordinates of its vertices are equal, and is horizontal when the second coordinates are equal. A *bend* in a path is a pair of consecutive edges $e_1, e_2$ of that path, such that the directions of $e_1$ and $e_2$ are different. When two edges $e_1$ and $e_2$ form a bend, they are called *bend edges*. A *segment* is a set of consecutive edges with no bends. Two paths are said to be *edge-intersecting*, or simply *intersecting* if they share at least one edge. Throughout the paper, any time we say that two paths intersect, we mean that they edge-intersect. If every path in a representation of a graph $G$ has at most $k$ bends, we say that this graph $G$ has a $B_k$-*EPG* representation. When $k = 1$ we say that this is a *single bend* representation.

In this paper, we study the Helly-$B_k$-EPG graphs. First, we show that every graph admits an EPG representation that is Helly, and present a characterization of Helly-$B_1$-EPG representations. Besides, we relate Helly-$B_1$-EPG graphs with L-shaped graphs, a natural family of subclasses of $B_1$-EPG. Finally, we prove that recognizing Helly-$B_k$-EPG graphs is in NP, for every fixed $k$. Besides, we show that recognizing Helly-$B_1$-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

The rest of the paper is organized as follows. In Section 2, we present some preliminary results, we show that every graph is a Helly-EPG graph, present a characterization of Helly-$B_1$-EPG representations, and relate Helly-$B_1$ EPG with L-shaped graphs. In Section 3, we discuss the NP-membership of HELLY-$B_k$ EPG RECOGNITION. In Section 4, we present the NP-completeness of recognizing Helly-$B_1$-EPG graphs.

# 2 Preliminaries

The study starts with the following lemma.

**Lemma 1** (Golumbic et al. (2009))**.** *Every graph is an EPG graph.*

We show that this result extends to Helly-EPG graphs.

**Lemma 2.** *Every graph is a Helly-EPG graph.*

**Proof:** Let $G$ be a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and $\mu$ maximal cliques $C_1, C_2, \ldots, C_\mu$. We construct a Helly-EPG representation of $G$ using a $\mu + 1 \times \mu + 1$ grid $Q$. Each maximal clique $C_i$ of $G$ is mapped to an edge of $Q$ as follow:

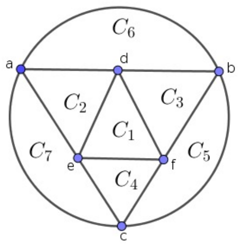
- if $i$ is even then the maximal clique $C_i$ is mapped to the edge in column $i$ between rows $i$ and $i + 1$;
- if $i$ is odd then the maximal clique $C_i$ is mapped to the edge in row $i$ between columns $i$ and $i + 1$.

The following describes a descendant-stair-shaped construction for the paths.

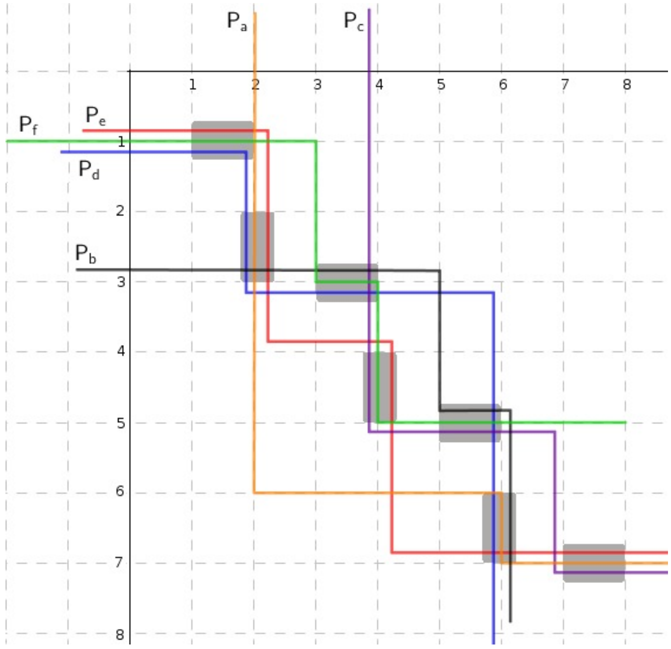
Let $v_l \in V(G)$ and $C_i$ be the first maximal clique containing $v_l$ according to the increasing order of their indices. If $i$ is even (resp. odd) the path $P_l$ starts in column $i$ (resp. in row $i$), in the point $(i, i)$. Then $P_l$ extends to at least the point $(i + 1, i)$ (resp. $(i, i + 1)$) proceeding to the until the row (resp. column) corresponding to next maximal clique of the sequence containing $v_l$, we say $C_j$. At this point, we bend $P_l$, which goes to the point $(j, j)$ and repeat the process previously described. Figure 3 shows the Helly-EPG representation of the octahedral graph $O_3$, according to the construction previously described.

By construction, each path travels only rows and columns corresponding with maximal cliques containing its respective vertex. And, every path crosses the edges of the grid to which your maximal cliques were mapped. Thus, the previously described construction results in an EPG representation of $G$, which is Helly since every set $\mathcal{P}$ of paths representing a maximal clique has at least one edge in its core. □

**Definition 3.** *The* Helly-bend number *of a graph $G$, denoted by $b_H(G)$, is the smallest $k$ for which $G$ is a Helly-$B_k$-EPG graph. Also, the bend number of a graph class $\mathcal{C}$ is the smallest $k$ for which all graphs in $\mathcal{C}$ have a $B_k$-EPG representation.*




(a) The octahedral graph $O_3$




(b) A Helly-EPG representation of the graph $O_3$

**Fig. 3:** Helly-EPG representation of the graph $O_3$ according to the construction of Lemma 2. The paths have been extended to the first/last row or column to improve the presentation.

**Corollary 4.** *For every graph $G$ containing $\mu$ maximal cliques, it holds that $b_H(G) \leq \mu - 1$.*

**Proof:** From the construction presented in Lemma 2, it follows that any graph admits a Helly-EPG representation where its paths have a descendant-stair shape. Since the number of bends in such a stair-shaped path is the number of maximal cliques containing the represented vertex minus one, it holds that $b_H(G) \leq \mu - 1$ for any graph $G$. □

Next, we examine the $B_1$-EPG representations of a few graphs that we employ in our constructions.

Given an EPG representation of a graph $G$, for any grid edge $e$, the set of paths containing $e$ is a clique in $G$; such a clique is called an edge-clique. A claw in a grid consists of three grid edges meeting at a grid point. The set of paths that contain two of the three edges of a claw is a clique; such a clique is called a claw-clique, see Golumbic et al. (2009). Fig. 1 illustrates an edge-clique and a claw-clique.

**Lemma 5** (Golumbic et al. (2009))**.** *Consider a $B_1$-EPG representation of a graph $G$. Every clique in $G$ corresponds to either an edge-clique or a claw-clique.*

Next, we present a characterization of Helly-$B_1$-EPG representations.

**Lemma 6.** *A $B_1$-EPG representation of a graph $G$ is Helly if and only if each clique of $G$ is represented by an edge-clique, i.e., it does not contain any claw-clique.*

**Proof:** Let $R$ be a $B_1$-EPG representation of a graph $G$. It is easy to see that if $R$ has a claw-clique, it does not satisfy the Helly property. Now, suppose that $R$ does not satisfy the Helly property. Thus it has a set $\mathcal{P}$ of pairwise intersecting paths having no common edge. Note that the set $\mathcal{P}$ represents a clique of $G$, and by Lemma 5, every clique in $G$ corresponds to either an edge-clique or a claw-clique. Since $\mathcal{P}$ represents a clique, but its paths have no common edge, then it has a claw-clique. □

Now, we consider EPG representations of $C_4$.

**Definition 7.** *Let $Q$ be a grid and let $(a_1, b)$, $(a_2, b)$, $(a_3, b)$, $(a_4, b)$ be a 4-star as depicted in Figure 4(a). Let $\mathcal{P} = \{P_1, \ldots, P_4\}$ be a collection of distinct paths each containing exactly two edges of the 4-star.*

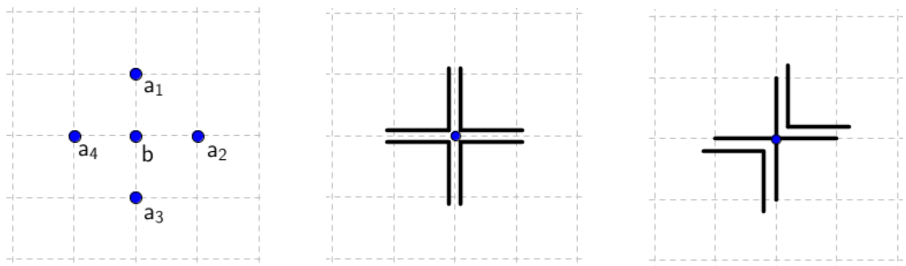
- *A* true pie *is a representation where each $P_i$ of $\mathcal{P}$ forms a bend in $b$.*
- *A* false pie *is a representation where two of the paths $P_i$ do not contain bends, while the remaining two do not share an edge.*

Fig. 4 illustrates true pie and false pie representations of a $C_4$.

**Definition 8.** *Consider a rectangle of any size with 4 corners at points $(x_1, y_1)$; $(x_2, y_1)$; $(x_2, y_2)$; $(x_1, y_2)$, positioned as in Fig. 5(a).*

- *A* frame *is a representation containing 4 paths $\mathcal{P} = \{P_1, \ldots, P_4\}$, each having a bend in a different corner of a rectangle, and such that the sub-paths $P_1 \cap P_2, P_1 \cap P_3, P_2 \cap P_4, P_3 \cap P_4$ share at least one edge. While $P_1 \cap P_4$ and $P_2 \cap P_3$ are empty sets.*
- *A square-frame is a frame where $P_1$, $P_2$, $P_3$ and $P_4$ have respectively point of bend $(x_1, y_1)$, $(x_2, y_1)$, $(x_1, y_2)$ and $(x_2, y_2)$, and are of the shape ∟, ⌋, ⌜ and ⌝. (see Fig.5)*

Fig. 5 illustrates some frame representations of a $C_4$.

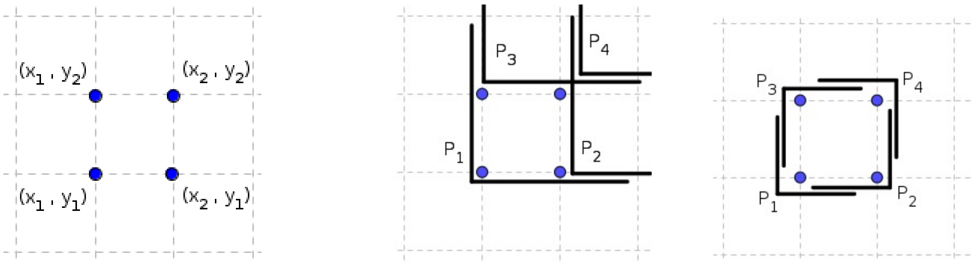



(a) 4-star in grid (b) True pie (c) False pie

**Fig. 4:** $B_1$-EPG representation of the induced cycle of size 4 as pies with emphasis in center $b$

**Lemma 9** (Golumbic et al. (2009))**.** *Every $C_4$ that is an induced subgraph of a graph G corresponds, in any representation, to a true pie, a false pie, or a frame.*

The following is a claim of Heldt et al. (2014b) which a reasoning can be found in Asinowski and Suk (2009).




(a) Points of the coordinates of bends of a frame (b) An example of a frame (c) A square-frame

**Fig. 5:** $B_1$-EPG representation of the induced cycle of size 4 as frame

**Lemma 10** (Heldt et al. (2014a) and Asinowski and Suk (2009))**.** *In every single bend representation of a $K_{2,4}$, the path representing each vertex of the largest part has its bend in a false pie.*

By creating four $K_{2,4}$ and identifying a vertex of the largest part of each one to a distinct vertex of a $C_4$, we construct the graph called bat graph (see Fig 6). Regarding to such a graph, the following holds.

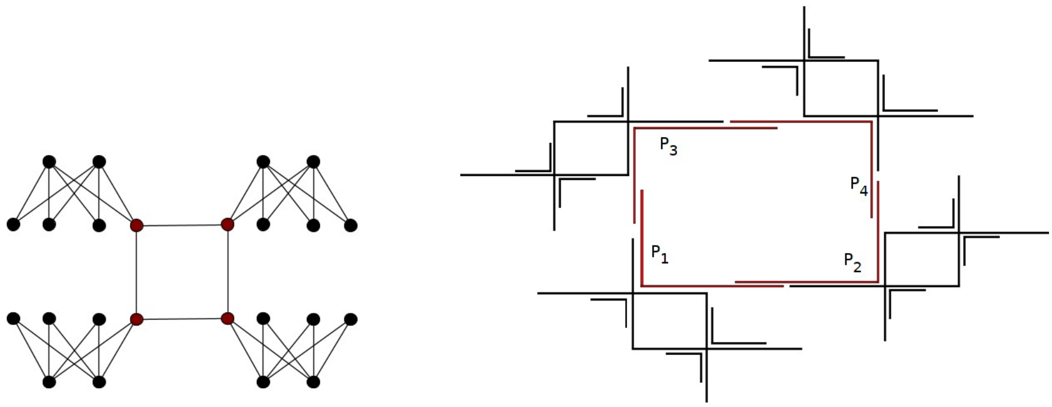



**Fig. 6:** A bat graph $G$ and a Helly-$B_1$-EPG representation of $G$.

**Corollary 11.** *In every single bend representation of the bat graph, G presented in Fig. 6, the $C_4$ that is a transversal of all $K_{2,4}$ is represented by a square-frame.*

**Proof:** By Lemma 10, it follows that in every single bend representation of the bat graph, each path representing a vertex of the $C_4$ (transversal to all $K_{2,4}$) has its bend in a false pie in which paths represent vertices of a $K_{2,4}$ (Fig. 7 illustrates a $B_1$-EPG representation of a $K_{2,4}$). Thus, the intersection of two paths representing vertices of this $C_4$ does not contain any edge incident to a bend point of such paths, which implies that such a $C_4$ must be represented by a frame (see Lemma 9). Note that for each path of the frame, we have four possible shapes (⌞, ⌟, ⌜, and ⌝). Let $P_1$ be the path having the bottom-left bend point, $P_2$ be the path having the bottom-right bend point, $P_3$ be the path having the top-left bend point and $P_4$ be the path having the top-right bend point. Note that to prevent $P_2$ and $P_3$ from containing edges incident at the bend point of $P_1$, the only shape allowed for $P_1$ is ⌞. Similarly, the only shape allowed for $P_2$ is ⌟ as well as for $P_3$ is ⌜ and for $P_4$ is ⌝. Thus, the $C_4$ is represented by a square-frame. □

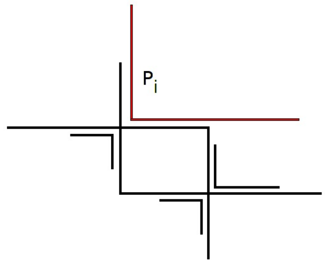



**Fig. 7:** Helly-$B_1$-EPG representation of a $K_{2,4}$.

**Definition 12.** *A $B_k$-EPG representation is* minimal *when its set of edges does not properly contain another $B_k$-EPG representation.*

The *octahedral* graph is the graph containing 6 vertices and 12 edges, depicted in Figure 8(a). Next, we consider representations of the octahedral graph.

The next lemma follows directly from the discussion presented in Heldt et al. (2014b).

**Lemma 13.** *Every minimal $B_1$-EPG representation of the octahedral graph $O_3$ has the same shape.*

**Proof:** Note that the octahedral graph $O_3$ has an induced $C_4$ such that the two vertices of the octahedral graph that are not in such a cycle are false twins whose neighborhood contains the vertices of the induced $C_4$.

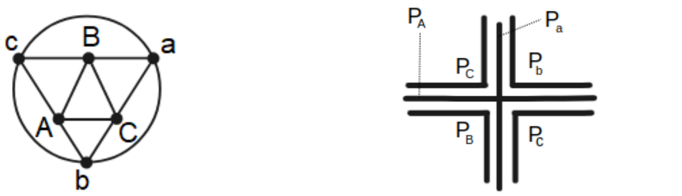
If in an EPG representation of the graph $O_3$ such a $C_4$ is represented as a frame, then no single bend path can simultaneously intersect the four paths representing the vertices of the induced $C_4$. Therefore, we conclude that the frame structure cannot be used to represent such a $C_4$ in a $B_1$-EPG representation of the $O_3$. Now, take a $B_1$-EPG representation of such a $C_4$ shaped as a true pie or false pie. By adding the paths representing the false twin vertices, which are neighbors of all vertices of the $C_4$, in both cases (from a true or false pie), we obtain representations with the shape represented in Fig. 8(b). □

## 2.1 Subclasses of $B_1$-EPG graphs

By Lemma 13, every minimal $B_1$-EPG representation of the octahedral graph $O_3$ has the same shape, as depicted in Fig. 8(b). Since in any representation of the graph $O_3$ there is always a triple of paths that do not satisfy the Helly property, paths $P_a, P_b$ and $P_c$ in the case of Fig. 8(b), it holds that $O_3 \notin$ Helly-$B_1$ EPG, which implies that the class of Helly-$B_1$-EPG graphs is a proper subclass of $B_1$-EPG.

Also, $B_0$-EPG and Helly-$B_0$-EPG graphs coincide. Hence, Helly-$B_0$ EPG can be recognized in polynomial time, see Booth and Lueker (1976).

In a $B_1$-EPG representation of a graph, the paths can be of the following four shapes: ⌞, ⌟, ⌜ and ⌝. Cameron et al. (2016) studied $B_1$-EPG graphs whose paths on the grid belong to a proper subset of the




(a) The octahedral graph $O_3$ (b) $B_1$-EPG representation of the graph $O_3$

**Fig. 8:** The octahedral graph $O_3$ graph and its $B_1$-EPG representation

four shapes. If $S$ is a subset of $\{\llcorner, \lrcorner, \ulcorner, \urcorner\}$, then $[S]$ denotes the class of graphs that can be represented by paths whose shapes belong to $S$, where zero-bend paths are considered to be degenerate $\llcorner$'s. They consider the natural subclasses of $B_1$-EPG: $[\llcorner]$, $[\llcorner, \ulcorner]$, $[\llcorner, \urcorner]$ and $[\llcorner, \ulcorner, \urcorner]$, all other subsets are isomorphic to these up to 90 degree rotation. Cameron et al. (2016) showed that recognizing each of these classes is NP-complete.

The following shows how these classes relate to the class of Helly-$B_1$-EPG graphs.

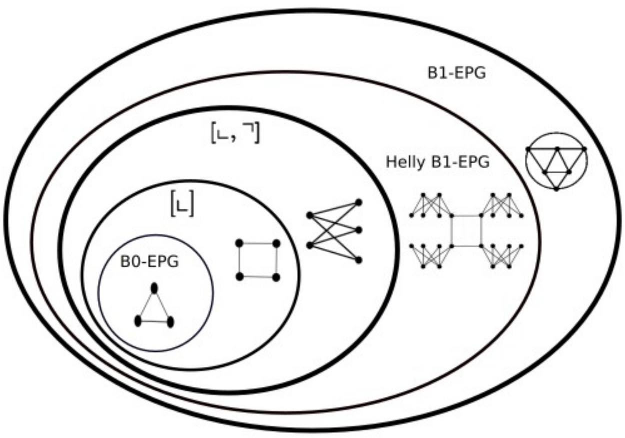



**Fig. 9:** Hierarchical diagram of some EPG classes

**Theorem 14.** $[\llcorner] \subsetneq [\llcorner, \urcorner] \subsetneq$ *Helly-$B_1$ EPG, and Helly-$B_1$ EPG is incomparable with* $[\llcorner, \ulcorner]$ *and* $[\llcorner, \ulcorner, \urcorner]$.

**Proof:** Cameron et al. (2016) showed that $[\llcorner] \subsetneq [\llcorner, \urcorner]$. Also, it is easy to see that $\llcorner$'s and $\urcorner$'s cannot form a claw-clique, thus, by Lemma 6, it follows that $[\llcorner, \urcorner] \subseteq$ Helly-$B_1$ EPG. In order to observe that $[\llcorner, \urcorner]$ is a proper subclass of Helly-$B_1$ EPG, it is enough to analyze the bat graph (see Fig. 6): by Corollary 11 follows that any $B_1$-EPG representation of a bat graph contains a square-frame, thus it is not in $[\llcorner, \urcorner]$. In addition, the bat graph is bipartite which implies that any $B_1$-EPG representation of that graph does not contain claw-cliques and therefore is Helly.

Now, it remains to show that Helly-$B_1$ EPG is incomparable with [⌞, ⌜] and [⌞, ⌜, ⌝]. Again, since any $B_1$-EPG representation of a bat graph contains a square-frame, bat graph is a Helly-$B_1$-EPG graph that is not in [⌞, ⌜, ⌝]. On the other hand, the $S_3$ (3-sun) is a graph in [⌞, ⌜] such that any of its $B_1$-EPG representations have a claw-clique, see Observation 7 in Cameron et al. (2016). Therefore, $S_3$ is a graph in [⌞, ⌜] that is not Helly-$B_1$ EPG. □

Figure 9 depicts example of graphs of the classes $B_0$-EPG, [⌞], [⌞, ⌝], Helly-$B_1$ EPG, and $B_1$-EPG that distinguish these classes.

It is known that recognizing [⌞], [⌞, ⌝], and $B_1$-EPG are NP-complete while recognizing $B_0$-EPG and EPG graphs can be done in polynomial time (c.f. Booth and Lueker (1976), Heldt et al. (2014b), and Cameron et al. (2016)).

In this paper, we show that it is NP-complete to recognize Helly-$B_1$-EPG graphs.

# 3 Membership in NP

The HELLY-$B_k$ EPG RECOGNITION problem can be formally described as follows.

| HELLY-$B_k$ EPG RECOGNITION | |
|---|---|
| *Input*: | A graph $G$ and an integer $k \leq \lvert V(G) \rvert^c$, for some fixed $c$. |
| *Goal:* | Determine if there is a set of $k$-bend paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ in a grid $Q$ such that:<br>• $u, v \in V(G)$ are adjacent in $G$ if only if $P_u, P_v$ share an edge in $Q$; and<br>• $\mathcal{P}$ satisfies the Helly property. |

A (positive) certificate for the HELLY-$B_k$ EPG RECOGNITION consists of a grid $Q$, a set $\mathcal{P}$ of $k$-bend paths of $Q$, which is in one-to-one correspondence with the vertex set $V(G)$ of $G$, such that, for each pair of distinct paths $P_i, P_j \in \mathcal{P}, P_i \cap P_j \neq \emptyset$ if and only if the corresponding vertices are adjacent in $G$. Furthermore, $\mathcal{P}$ satisfies the Helly property.

The following are key concepts that make it easier to control the size of an EPG representation. A *relevant edge* of a path in a $B_k$-EPG representation is either an extremity edge or a bend edge of the path. Note that each path with at most $k$ bends can have up to $2(k+1)$ relevant edges, and any $B_k$-EPG representation contains at most $2|\mathcal{P}|(k+1)$ distinct relevant edges.

To show that there is a non-deterministic polynomial-time algorithm for HELLY-$B_k$ EPG RECOGNITION, it is enough to consider as certificate a $B_k$-EPG representation $R$ containing a collection $\mathcal{P}$ of paths, $|\mathcal{P}| = |V(G)|$, such that each path $P_i \in \mathcal{P}$ is given by its set of relevant edges along with the relevant edges, that intersects $P_i$, of each path $P_j$ intersecting $P_i$, where $P_j \in \mathcal{P}$. The relevant edges for each path are given in the order that they appear in the path, to make straightforward checking that the edges correspond to a unique path with at most $k$ bends. This representation is also handy for checking that the paths form an intersection model for $G$.

To verify in polynomial time that the input is a positive certificate for the problem, we must assert the following:

(i) The sequence of relevant edges of a path $P_i \in \mathcal{P}$ determines $P_i$ in polynomial time;

(ii) Two paths $P_i, P_j \in \mathcal{P}$ intersect if and only if they intersect in some relevant edge;

(iii) The set $\mathcal{P}$ of relevant edges satisfies the Helly property.

The following lemma states that condition (i) holds.

**Lemma 15.** *Each path $P_i$ can be uniquely determined in polynomial time by the sequence of its relevant edges.*

**Proof:** Consider the sequence of relevant edges of some path $P_i \in \mathcal{P}$. Start from an extremity edge of $P_i$. Let $t$ be the row (column) containing the last considered relevant edge. The next relevant edge $e'$ in the sequence, must be also contained in row (column) $t$. If $e'$ is an extremity edge, the process is finished, and the path has been determined. It contains all edges between the considered relevant edges in the sequence. Otherwise, if $e'$ is a bend edge, the next relevant edge is the second bend edge $e''$ of this same bend, which is contained in some column (row) $t'$. The process continues until the second extremity edge of $P_i$ is located.

With the above procedure, we can determine in $\mathcal{O}(k \cdot |V(G)|)$ time, whether path $P_i$ contains any given edge of the grid $Q$. Therefore, the sequence of relevant edges of $P_i$ uniquely determines $P_i$. □

Next, we assert property (ii).

**Lemma 16.** *Let $\mathcal{P}$ be the set of paths in a $B_k$-EPG representation of G, and let $P_1, P_2 \in \mathcal{P}$. Then $P_1$, $P_2$ are intersecting paths if and only if their intersection contains at least one relevant edge.*

**Proof:** Assume that $P_1, P_2$ are intersecting, and we show they contain a common relevant edge. Without loss of generality, suppose $P_1, P_2$ intersect at row *i* of the grid, in the $B_k$-EPG representation $R$. The following are the possible cases that may occur:

- **Case 1:** Neither $P_1$ nor $P_2$ contain bends in row *i*.

  Then $P_1$ and $P_2$ are entirely contained in row *i*. Since they intersect, either $P_1, P_2$ overlap, or one of the paths contains the other. In any of these situations, they intersect in a common extremity edge, which is a relevant edge.

- **Case 2:** $P_1$ does not contain bends in *i*, but $P_2$ does.

  If some bend edge of $P_2$ also belongs to $P_1$, then $P_1, P_2$ intersect in a relevant edge. Otherwise, since $P_1, P_2$ intersect, the only possibility is that the intersection contains an extremity edge of $P_1$ or $P_2$. Hence the paths intersect in a relevant edge.

- **Case 3:** Both $P_1$, $P_2$ contain bends in *i*

  Again, if the intersection occurs in some bend edge of $P_1$ or $P_2$, the lemma follows. Otherwise, the same situation as above must occur: $P_1, P_2$ must intersect in an extremity edge.

In any of the cases, $P_1$ and $P_2$ intersect in some relevant edge. □

The two previous lemmas let us check that a certificate is an actual $B_k$-EPG representation of a given graph $G$. The next lemma says we can also verify in polynomial time that the representation encoded in the certificate is a Helly representation. Fortunately, we do not need to check every subset of intersecting paths of the representation to make sure they have a common intersection.

**Lemma 17.** *Let $\mathcal{P}$ be a collection of paths encoded as a sequence of relevant edges that constitute a $B_k$-EPG representation of a graph $G$. We can verify in polynomial time if $\mathcal{P}$ has the Helly property.*

**Proof:** Let $T$ be the set of relevant edges of $\mathcal{P}$. Consider each triple $T_i$ of edges of $T$ . Let $P_i$ be the set of paths of $\mathcal{P}$ containing at least two of the edges in the triple $T_i$. By Gilmore's Theorem, see Berge and Duchet (1975), $\mathcal{P}$ has the Helly property if an only if the subset of paths $P_i$ corresponding to each triple $T_i$ has a non-empty intersection. By Lemma 16, it suffices to examine the intersections on relevant edges. Therefore a polynomial algorithm for checking if $\mathcal{P}$ has the Helly property could examine each of the subsets $P_i$, and for each relevant edge $e$ of a path in $P_i$, to compute the number of paths in $P_i$ that contain $e$. Then $\mathcal{P}$ has the Helly property if and only if for every $P_i$, there exists some relevant edge that is present in all paths in $P_i$, yielding a non-empty intersection. □

**Corollary 18.** *Let $\mathcal{P}'$ be a set a pairwise intersecting paths in a Helly-$B_k$-EPG representation of a graph $G$. Then the intersection of all paths of $\mathcal{P}'$ contains at least one relevant edge.*

Note that the property described in Corollary 18 is a consequence of Gilmore's Theorem, see Berge and Duchet (1975), and it applies only to representations that satisfy Helly's property.

From Corollary 18, the following theorem concerning the Helly-bend number of a graph holds.

**Theorem 19.** *For every graph $G$ containing $n$ vertices and $\mu$ maximal cliques, it holds that*

$$\frac{\mu}{2n} - 1 \leq b_H(G) \leq \mu - 1.$$

**Proof:** The upper bound follows from Corollary 4. For the lower bound first notice that each path with at most $k$ bends can have up to $2(k+1)$ relevant edges, and any $B_k$-EPG representation with a set of paths $\mathcal{P}$ contains at most $2|\mathcal{P}|(k+1)$ distinct relevant edges. Now, let $G$ be a graph with $n$ vertices, $\mu$ maximal cliques, and $b_H(G) = k$. From Corollary 18, it follows that in a Helly-$B_k$-EPG representation of $G$ every maximal clique of $G$ contains at least one relevant edge. By maximality, two distinct maximal cliques cannot share the same edge-clique. Thus, in a Helly-$B_k$-EPG representation of $G$ every maximal clique of $G$ contains at least one distinct relevant edge, which implies that $\mu \leq 2n(k+1)$, so $\frac{\mu}{2n} - 1 \leq b_H(G)$. □

**Lemma 20.** *Let $G$ be a (Helly-)$B_k$-EPG graph. Then $G$ admits a (Helly-)$B_k$-EPG representation on a grid of size at most $4n(k+1) \times 4n(k+1)$.*

**Proof:** Let $R$ be a $B_k$-EPG representation of a graph $G$ on a grid $Q$ with the smallest possible size. Let $\mathcal{P}$ be the set of paths of $R$. Note that $|\mathcal{P}| = n$. A counting argument shows that there are at most $2|\mathcal{P}|(k+1)$ relevant edges in $R$. If $Q$ has a pair of consecutive columns $c_i, c_{i+1}$ neither of which contains relevant edges of $R$, and such that there is no relevant edge crossing from $c_i$ to $c_{i+1}$, then we can contract each edge crossing from $c_i$ to $c_{i+1}$ into single vertices so as to obtain a new $B_k$-EPG representation of $G$ on a smaller grid, which is a contradiction. An analogous argument can be applied to pairs of consecutive rows of the grid. Therefore the grid $Q$ is such that each pair of consecutive columns and consecutive rows of $Q$ has at least one relevant edge of $R$ or contains a relevant edge crossing it. Since $Q$ is the smallest possible grid for representing $G$, then the first row and the first column of $Q$ must contain at least one point belonging to some relevant edge of $R$. Thus, if $G$ is $B_k$-EPG then it admits a $B_k$-EPG representation on a grid of size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. Besides, by Corollary 18, it holds that the contraction operation previously described preserves the Helly property, if any. Hence, letting $R$ be a Helly-$B_k$-EPG representation of a graph $G$ on a grid $Q$ with the smallest possible size it holds that $Q$ has size at most $4|\mathcal{P}|(k+1) \times 4|\mathcal{P}|(k+1)$. □

Given a graph $G$ with $n$ vertices and an EPG representation $R$, it is easy to check in polynomial time with respect to $n + |R|$ whether $R$ is a $B_k$-EPG representation of $G$. By Lemma 20, if $G$ is a $B_k$-EPG graph then there is a positive certificate (an EPG representation) $R$ of polynomial size with respect to $k + n$ to the question "$G \in B_k$-EPG?". Therefore, Corollary 21 holds.

**Corollary 21.** *Given a graph G and an integer $k \geq 0$, the problem of determining whether G is a $B_k$-EPG graph is in NP, whenever k is bounded by a polynomial function of $|V(G)|$.*

At this point, we are ready to demonstrate the NP-membership of HELLY-$B_k$ EPG RECOGNITION.

**Theorem 22.** HELLY-$B_k$ EPG RECOGNITION *is in NP.*

**Proof:** By Lemma 20 and the fact that $k$ is bounded by a polynomial function of $|V(G)|$, it follows that the collection $\mathcal{P}$ can be encoded through its relevant edges with $n^{\mathcal{O}(1)}$ bits.

Finally, by Lemmas 15, 16 and 17, it follows that one can verify in polynomial-time in the size of $G$ whether $\mathcal{P}$ is a family of paths encoded as a sequence of relevant edges that constitute a Helly-$B_k$-EPG representation of a graph $G$. □

# 4 NP-hardness

Now we will prove that HELLY-$B_1$ EPG RECOGNITION is NP-complete. For this proof, we follow the basic strategy described in the prior hardness proof of Heldt et al. (2014b). We set up a reduction from POSITIVE (1 IN 3)-3SAT defined as follows:

| POSITIVE (1 IN 3)-3SAT | |
|---|---|
| *Input*: | A set $X$ of positive variables; a collection $C$ of clauses on $X$ such that for each $c \in C$, $|c| = 3$. |
| *Goal:* | Determine if there is an assignment of values to the variables in $X$ so that every clause in $C$ has exactly one true literal. |

POSITIVE (1 IN 3)-3SAT is a well-known NP-complete problem (see Garey and Johnson (1979), problem [L04], page 259). Also, it remains NP-complete when the incidence graph of the input CNF (Conjunctive Normal Form) formula is planar, see Mulzer and Rote (2008).

Given a formula $F$ that is an instance of POSITIVE (1 IN 3)-3SAT we will present a polynomial-time construction of a graph $G_F$ such that $G_F \in$ Helly-$B_1$ EPG if and only if $F$ is satisfiable. This graph will contain an induced subgraph $G_{C_i}$ with 12 vertices (called *clause gadget*) for every clause $C_i \in \mathcal{C}$, and an induced subgraph (*variable gadget*) for each variable $x_j$, containing a special vertex $v_j$, plus a *base gadget* with 55 additional vertices.

We will use a graph $H$ isomorphic to the graph presented in Figure 10, as a gadget to perform the proof. For each clause $C_i$ of $F$ of the target problem, we will have a *clause gadget* isomorphic to $H$, denoted by $G_{c_i}$.

The reduction of a formula $F$ from POSITIVE (1 IN 3)-3SAT to a particular graph $G_F$ (where $G_F$ has a Helly-$B_1$-EPG representation if only if $F$ is satisfiable) is given below.

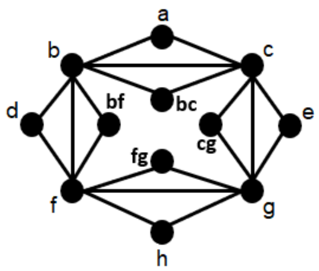



**Fig. 10:** The partial gadget graph $H$

**Definition 23.** *Let $F$ be a CNF-formula with variable set $\mathcal{X}$ and clause set $\mathcal{C}$ with no negative literals, in which every clause has exactly three literals. The graph $G_F$ is constructed as follows:*

1. *For each clause $C_i \in \mathcal{C}$ create a clause gadget $G_{C_i}$, isomorphic to graph $H$;*
2. *For each variable $x_j \in \mathcal{X}$ create a* variable vertex *$v_j$ that is adjacent to the vertex $a$, $e$, or $h$ of $G_{C_i}$, when $x_j$ is the first, second or third variable in $C_i$, respectively;*
3. *For each variable vertex $v_j$, construct a* variable gadget *formed by adding two copies of $H$, $H_1$ and $H_2$, and making $v_j$ adjacent to the vertices of the triangles $(a, b, c)$ in $H_1$ and $H_2$.*
4. *Create a vertex $V$, that will be used as a vertical reference of the construction, and add an edge from $V$ to each vertex $d$ of a clause gadget;*
5. *Create a bipartite graph $K_{2,4}$ with a particular vertex $T$ in the largest stable set. This vertex is nominated* true vertex. *Vertex $T$ is adjacent to all $v_j$ and also to $V$;*
6. *Create two graphs isomorphic to $H$, $G_{B1}$ and $G_{B2}$. The vertex $T$ is connected to each vertex of the triangle (a,b,c) in $G_{B1}$ and $G_{B2}$;*
7. *Create two graphs isomorphic to $H$, $G_{B3}$ and $G_{B4}$. The vertex $V$ is connected to each vertex of the triangle (a,b,c) in $G_{B3}$ and $G_{B4}$;*
8. *The subgraph induced by the set of vertices $\{V(K_{2,4}) \cup \{T, V\} \cup V(G_{B1}) \cup V(G_{B2}) \cup V(G_{B3}) \cup V(G_{B4})\}$ will be referred to as the* base gadget.

Figure 11 illustrates how this construction works on a small formula.

**Lemma 24.** *Given a satisfiable instance $F$ of* POSITIVE (1 IN 3)-3SAT, *the graph $G_F$ constructed from $F$ according to Definition 23 admits a Helly-$B_1$-EPG representation.*

**Proof:** We will use the true pie and false pie structures to represent the *clause gadgets $G_C$* (see Figure 12), but the construction could also be done with the frame structure without loss of generality.

The *variable gadgets* will be represented by structures as of Figure 13.

The *base gadget* will be represented by the structure of Figure 14.

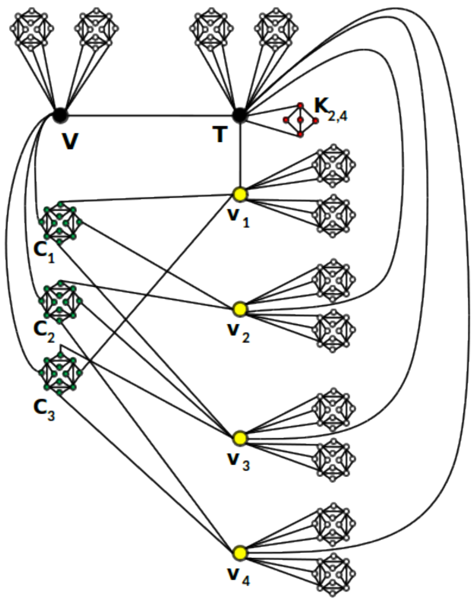


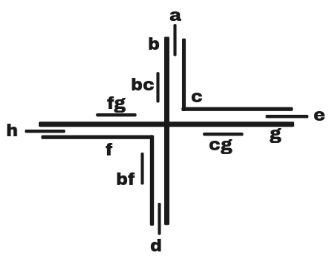

**Fig. 11:** The $G_F$ graph corresponding to formula $F = (x_1 + x_2 + x_3) \cdot (x_2 + x_3 + x_4) \cdot (x_3 + x_1 + x_4)$

It is easy to see that the representations of the clause gadgets, variable gadgets, and base gadgets are all Helly-$B_1$ EPG. Now, we need to describe how these representations can be combined to construct a single bend representation $R_{G_F}$.

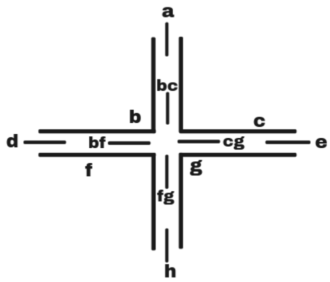
Given an assignment $A$ that satisfies $F$, we can construct a Helly-$B_1$-EPG representation $R_{G_F}$. First we will fix the representation structure of the base gadget in the grid to guide the single bend representation, see Figure 14. Next we will insert the variable gadgets with the following rule: if the variable $x_i$ related to the path $P_{v_i}$ had assignment *True*, then the adjacency between the path $P_{v_i}$ with $P_T$ is horizontal, and vertical otherwise. For example, for an assignment $A = \{x_1 = False; x_2 = False; x_3 = True; x_4 = False\}$ to variables of the formula $F$ that generated the gadget $G_F$ of Figure 11, it will give us a single bend representation (base gadget + variables gadget) according to the Figure 15(a).

When a formula $F$ of POSITIVE (1-IN-3)-3SAT has clauses whose format of the assignment is $(False, True, False)$ or $(False, False, True)$ then we will use false pie to represent these clauses. When the clause has format $(True, False, False)$, we will use true pie to represent this clause (the use of true pie in the last case is only to illustrate that the shape of the pie does not matter in the construction). To insert a *clause gadget* $G_C$, we introduce a horizontal line $l_h$ in the grid between the horizontal rows used by the paths for the two false variables in $C$. Then we connect the path $P_{d_{c_i}}$ of $G_{C_i}$ to $P_V$ vertically using the bend of $P_{d_{c_i}}$. We introduce a vertical line $l_v$ in the grid, between the vertical line of the grid used by $P_V$ and the path to the true variable in $C_i$, *i.e.* between $P_V$ and the path of the true variable $x_j \in C_i$. At the point where $l_h$ and $l_v$ cross, to insert the center of the *clause gadget* as can be seen in Figure 15(b). The complete construction of this single bend representation for the $G_F$ can be seen in Figure 16.

Note that when we join all these representations of gadgets that form $R_{G_F}$, we do not increase the




(a) Based in false pie




(b) Based in true pie

**Fig. 12:** Single bend representations of a clause gadget isomorphic to graph $H$

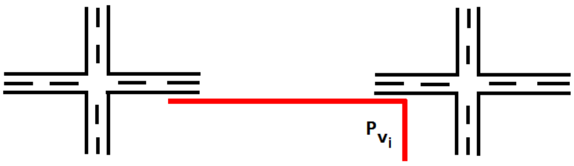



**Fig. 13:** Single bend representation of a variable gadget

number of bends. Then the representation necessarily is $B_1$-EPG. Let us show that it satisfies the Helly property.

A simple way to check that $R_{G_F}$ satisfies the Helly property is to note that the particular graph $G_F$ never forms triangles between variable, clause, and base gadgets. Thus, any triangle of $G_F$ is inside a variable, clause, or base gadget. As we only use Helly-$B_1$-EPG representations of such gadgets, $R_{G_F}$ is a Helly-$B_1$-EPG representation of $G_F$. □

Now, we consider the converse. Let $R$ be a Helly-$B_1$-EPG representation of $G_F$.

**Definition 25.** *Let $H$ be the graph shown in Figure 10, such that the 4-cycle $H[\{b, c, f, g\}]$ corresponds*

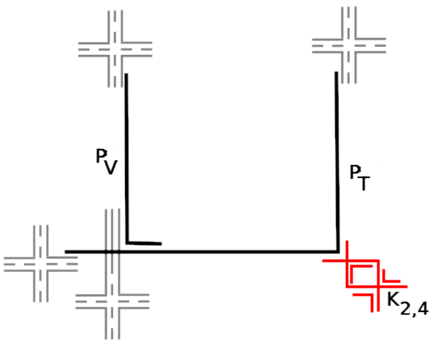



**Fig. 14:** Single bend representation of the base gadget

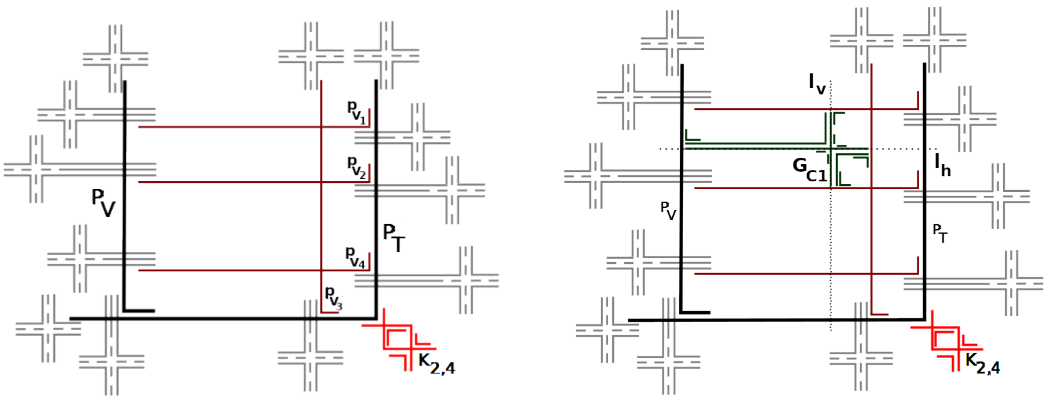
*in $R$ to a false pie or true pie, then:*

- *the* center *is the unique grid-point of this representation which is contained in every path representing 4-cycle $\{b, c, f, g\}$;*
- *a* central ray *is an edge-intersection between two of the paths corresponding to vertices $b, c, f, g$, respectively.*

Note that every $B_1$-EPG representation of a $C_4$ satisfies the Helly property, see Lemma 9, and triangles have $B_1$-EPG representations that satisfy the Helly property, *e.g.* the one shown in Figure 1(b). The graph $H$ is composed by a 4-cycle $C_4^H = H[b, c, f, g]$ and eight cycles of size 3.

As $C_4^H$ has well known representations (see in Lemma 9), then we can start drawing the Helly-$B_1$-EPG representation of $H$ from these structures. Figure 17 shows possible representations for $H$.

If $C_4^H$ is represented by a pie, then the paths $P_b, P_c, P_f, P_g$ share the center of the representation. On the other hand, if $C_4^H$ is represented by a frame, then the bends of the four paths correspond to the four distinct corners of a rectangle, *i.e.* all paths representing the vertices of $C_4^H$ have distinct bend points, see Golumbic et al. (2009).




(a) Representation with omitted clause gadgets

(b) Representation with $G_{C_1}$ associated with the clause $(x_1 + x_2 + x_3)$ in highlighted

**Fig. 15:** Single bend representation of the base and variables gadgets associated with the assignment $x_1 = False, x_2 = False, x_3 = True, x_4 = False$

Next, we examine the use of the frame structure.

**Proposition 26.** *In a frame-shaped $B_1$-EPG representation of a $C_4$, every path $P_i$ that represents a vertex of the $C_4$ intersects exactly two other paths $P_{i-1}$ and $P_{i+1}$ of the frame so that one of the intersections is horizontal and the other is vertical.*

**Proposition 27.** *Given a Helly-$B_1$-EPG representation of a graph $G$ that has an induced $C_4$ whose representation is frame-shaped. If there is a vertex $v$ of $G$, outside the $C_4$, that is adjacent to exactly two*

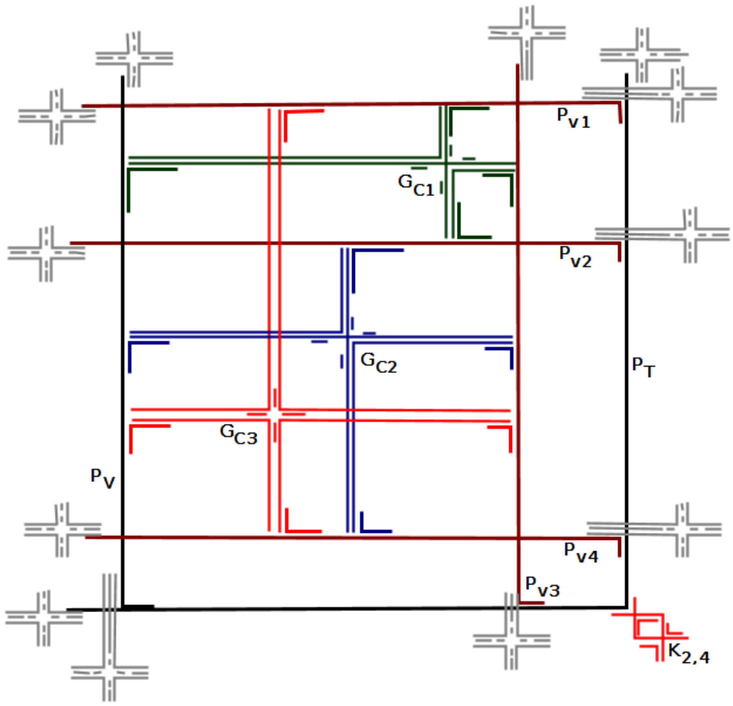



**Fig. 16:** Single bend representation of $G_F$

*consecutive vertices of this $C_4$, then the path representing $v$ shares at least one common edge-intersection with the paths representing both of these vertices.*

**Proof:** By assumption, $G$ has a triangle containing $v$ and two vertices of a $C_4$. Therefore the path representing $v$ shares at least one common edge intersecting with the paths representing these neighbors, otherwise the representation does not satisfy the Helly property. □

By Proposition 26 and Proposition 27 we can conclude that for every vertex $v_i \in V(H)$ such that $v_i \neq V(C_4^H)$, when we use a frame to represent the $C_4^H$, $P_{v_i}$ will have at least one common edge-intersection with the pair of paths representing its neighbors in $H$. Figure 17(c) presents a possible Helly-$B_1$-EPG representation of $H$. Note that we can apply rotations and mirroring operations while maintaining it as a Helly-$B_1$-EPG representation of $H$.

**Definition 28.** *In a frame-shaped single bend representation of a $C_4$ graph, the paths that represent consecutive vertices in the $C_4$ are called* consecutive paths *and the segment that corresponds to the intersection between two consecutive paths is called* side intersection*.*

**Lemma 29.** *In any minimal single bend representation of a graph isomorphic to $H$, there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal directions and the other two paths have vertical directions.*

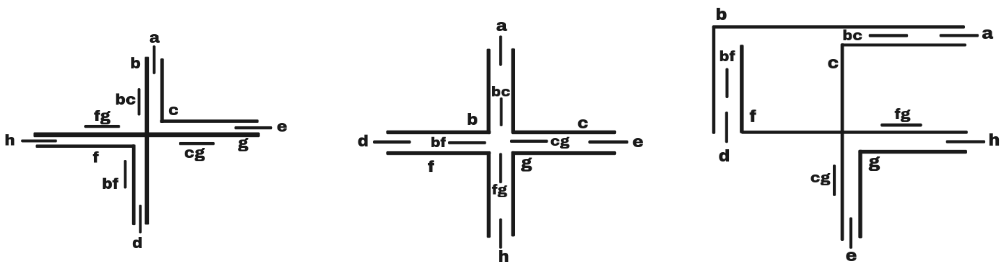
**Proof:** If the $C_4^H = [b, c, f, g]$ is represented by a true pie or false pie, then each path of $C_4^H$ shares two central rays with two other paths of $C_4^H$, where each central ray corresponds to one pair of consecutive vertices in $C_4^H$.

As the vertices $a, e, d$ and $h$ are adjacent to pairs of consecutive vertices in $C_4^H$ so the paths $P_a, P_e, P_d$ and $P_h$ have to be positioned in each one of the different central rays, 2 are horizontal and 2 are vertical.

If the $C_4^H$ is represented by a frame, then each path of the $C_4^H$ has a bend positioned in the corners of the frame. In the frame, the adjacency relationship of pairs of consecutive vertices in the $C_4^H$ is represented by the edge-intersection of the paths that constitute the frame. Thus, since a frame has two parts in the vertical direction and two parts in the horizontal direction, then there are two paths in $\{P_a, P_e, P_d, P_h\}$ that have horizontal direction and two that have vertical direction.

Note that no additional edge is needed on the different paths by the minimality of the representation. □

**Corollary 30.** *In any minimal single bend representation of a graph isomorphic to $H$, the following paths are on the same central ray or side intersection: $P_a$ and $P_{bc}$; $P_e$ and $P_{cg}$; $P_h$ and $P_{fg}$; $P_d$ and $P_{bf}$.*




(a) Based in false pie (b) Based in true pie (c) Based in frame

**Fig. 17:** Different single bend representations of the graph $H$ using a false pie (a), a true pie (b) and a frame (c) for representing $C_4^H$

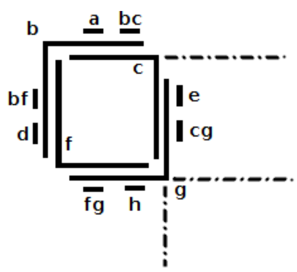



**Fig. 18:** A frame representation where the bend of dashed paths change directions

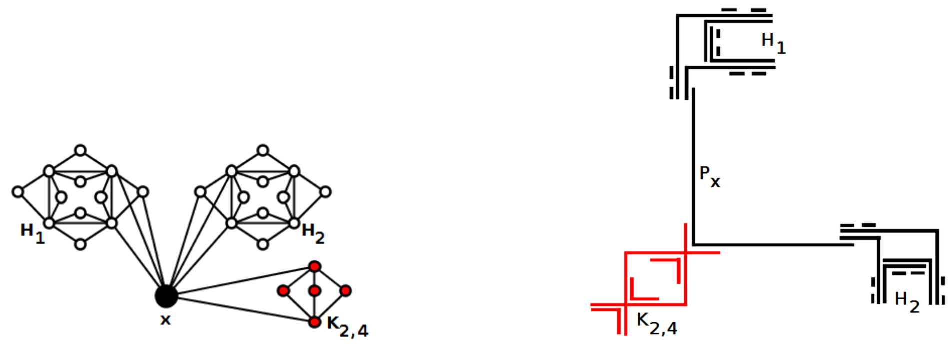
The following proposition helps us in the understanding of the NP-hardness proof.

**Proposition 31.** *In any Helly-$B_1$ representation of the graph $G'$, presented in Figure 19(a), the path $P_x$ has obstructed extremities and bends.*

**Proof:** Consider $G'$ consisting of a vertex $x$ together with two graphs, $H_1$ and $H_2$, isomorphic to $H$ and a bipartite graph $K_{2,4}$, such that: $x$ is a vertex of the largest stable set of the $K_{2,4}$; $x$ is adjacent to an induced cycle of size 3 of $H_1$, $C_3^{H_1}$ and to an induced cycle of size 3 of $H_2$, $C_3^{H_2}$, see Figure 19(a).

We know that the paths belonging to the largest stable set of a $K_{2,4}$ always will bend into a false pie, see Fact 10. Since $P_x$ is part of the largest stable set of the $K_{2,4}$, then $P_x$ has an *obstructed bend*, see Figure 19(b).

The vertex $x$ is adjacent to $C_3^{H_1}$ and $C_3^{H_2}$, so that its path $P_x$ intersects the paths representing them. But in a single bend representations of a graph isomorphic to $H$ there are pairs of paths that always are on some segment of a central ray or a side intersection, see Corollary 30, and the representation of $C_3^{H_1}$ ( similarly $C_3^{H_2}$) has one these paths. Therefore, there is an edge in the set of paths that represent $H_1$ ( similarly in $H_2$) that has a intersection of 3 paths representing $C_3^{H_1}$ (and $C_3^{H_2}$), otherwise the representation would not be Helly. There is another different edge in the same central ray or side intersection that contains three other paths and one of them is not in the set of paths $C_3^{H_1}$ ( similarly $C_3^{H_2}$). Thus in a single bend representation of $G'$, the paths that represent $C_3^{H_1}$ ( similarly $C_3^{H_2}$) must intersect in a bend edge or an extremity edge of $P_x$, because $P_x$ intersects only one of the paths that are on some central ray or side intersection where $C_3^{H_1}$ ( similarly $C_3^{H_2}$) is. As the bend of $G'$ is already obstructed by structure of $K_{2,4}$, then $H_1$ ( similarly in $H_2$) must be positioned at an extremity edge of $P_x$. This implies that $P_x$ has a condition of *obstructed extremities*, see Figure 19(b). □




(a) The graph $G'$ (b) A $B_1$-EPG representation of $G'$

**Fig. 19:** The sample of obstructed extremities and bend.

**Definition 32.** *We say that a segment $s$ is* internally contained *in a path $P_x$ if $s$ is contained in $P_x$, and it does not intersect a relevant edge of $P_x$.*

Some of the vertices of $G_F$ have highly constrained $B_1$-EPG representations. Vertex $T$ has its bend and both extremities obstructed by its neighbors in $G_{B1}$, $G_{B2}$ and in the $K_{2,4}$ subgraphs. Vertex $V$ and each variable vertex $v_i$ must have one of its segments internally contained in $T$, and also have its extremities and bends obstructed. Therefore, vertex $V$ and each variable vertex has only one segment each that can be used in an EPG representation to make them adjacent to the clause gadget. The direction of this segment, being either horizontal or vertical, can be used to represent the true or false value for the variable. The clause gadgets, on the other hand, are such that exactly two of its adjacencies to the variable vertices and

$V$ can be realized with a horizontal intersection, whereas the other two must be realized with a vertical intersection. If we consider the direction used by $V$ as a truth assignment, we get that exactly one of the variables in each clause will be true in any possible representation of $G_F$. Conversely, it is fairly straightforward to obtain a $B_1$-EPG representation for $G_F$ when given a truth assignment for the formula $F$. Therefore, Lemma 33 holds.

**Lemma 33.** *If a graph $G_F$, constructed according to Definition 23, admits a Helly-$B_1$-EPG representation, then the associated CNF-formula $F$ is a yes-instance of* POSITIVE (1 IN 3)-3SAT.

**Proof:** Suppose that $G_F$ has a Helly-$B_1$-EPG representation, $R_{G_F}$. From $R_{G_F}$ we will construct an assignment that satisfies $F$.

First, note that in every single bend representation of a $K_{2,4}$, the path of each vertex of the largest stable set, in particular, $P_T$ (in $R_{G_F}$), has bends contained in a false pie (see Lemma 10).

The vertex $T$ is adjacent to the vertices of a triangle of $G_{B1}$ and $G_{B2}$. As the $K_{2,4}$ is positioned in the bend of $P_T$, then in $R_{G_F}$ the representations of $G_{B1}$ and $G_{B2}$ are positioned at the extremities of $P_T$, see Proposition 4.3.

Without loss of generality assume that $P_V \cap P_T$ is a horizontal segment in $R_{G_F}$.

We can note in $R_{G_F}$ that: the number of paths $P_d$ with segment internally contained in $P_V$ is the number of clauses in $F$; the intersection between each $P_a, P_e, P_h$ in the gadget clause and each path $P_{v_j}$ indicates the variables composing the clause. Thus, we can assign to each variable $x_j$ the value *True* if the edge intersecting $P_{v_j}$ and $P_T$ is horizontal, and *False* otherwise.

In Lemma 29 it was shown that any minimal $B_1$-EPG representation of a clause gadget has two paths in $\{P_a, P_d, P_e, P_h\}$ with vertical direction and the other two paths have horizontal direction. Since $P_d$ intersects $P_V$, it follows that in a single bend representation of $G_F$, we must connect two of these to represent a false assignment, and exactly one will represent a true assignment. Thus, from $R_{G_F}$, we construct an assignment to $F$ such that every clause has exactly one variable with a true value. □

Recall that a $B_1$-EPG representation is Helly if and only if each clique is represented by an edge-clique (and not by a claw-clique). Thus, an alternative way to check whether a representation is Helly is to note that all cliques are represented as edge-cliques.

**Theorem 34.** HELLY-$B_1$ EPG RECOGNITION *is NP-complete.*

**Proof:** By Theorem 22, Lemma 24, Lemma 33. □

We say that a $k$-apex graph is a graph that can be made planar by the removal of $k$ vertices. A $d$-degenerate graph is a graph in which every subgraph has a vertex of degree at most $d$. Recall that POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of the input formula is planar, see Mulzer and Rote (2008). Thus, the following corollary holds.

**Corollary 35.** HELLY-$B_1$ EPG RECOGNITION *is NP-complete on* $2$*-apex and* $3$*-degenerate graphs.*

**Proof:** To prove that $G_F$ is 3-degenerate, we apply the $d$-degenerate graphs recognition algorithm, consisting of repeatedly removing the vertices of a minimum degree from the graph. Note that each vertex to be removed at each iteration of the algorithm always has a degree at most three, and therefore the graphs $G_F$ constructed according to Definition 23 is 3-degenerate.

Now, recall that POSITIVE (1 IN 3)-3SAT remains NP-complete when the incidence graph of $F$ is planar, see Mulzer and Rote (2008). Let $F$ be an instance of PLANAR POSITIVE (1 IN 3)-3SAT, we know that the incidence graph of the formula $F$ is planar. By using the planar embedding of the incidence graph, we can appropriately replace the vertices representing variables and clauses by variables gadgets and clauses gadgets. As each variable gadget, clause gadget, and base gadget are planar, then something not planar may have arisen only from the intersection that was made between them. As the incidence graph assures that there is a planar arrangement between the intersections of the variable gadgets and clause gadgets, then from that one can construct a graph $G_F$ such that the removal of $V$ and $T$ results into a planar graph, see Figura 20. Thus $G_F$ is 2-apex. □

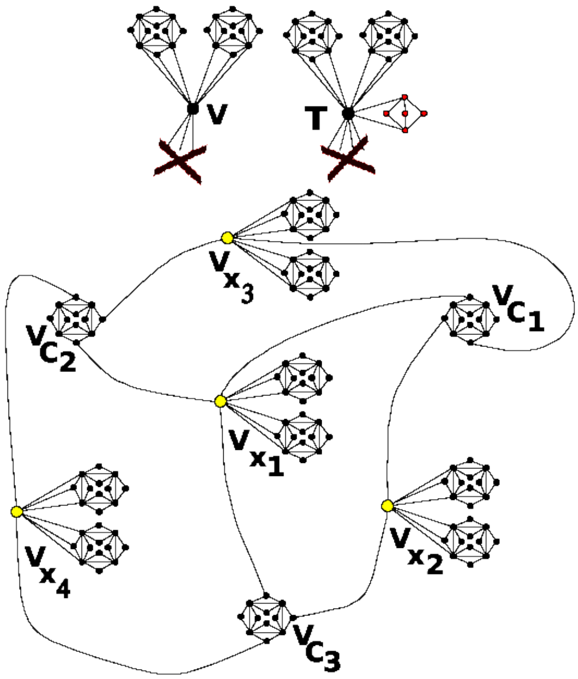



**Fig. 20:** Planar graph built from $F = (x_1 + x_2 + x_3) \cdot (x_1 + x_3 + x_4) \cdot (x_1 + x_2 + x_4)$, after removing $V$ and $T$.

# 5 Concluding Remarks

In this paper, we show that every graph admits a Helly-EPG representation, and $\frac{\mu}{2n} - 1 \leq b_H(G) \leq \mu - 1$. Besides, we relate Helly-$B_1$-EPG graphs with L-shaped graphs, a natural family of subclasses of $B_1$-EPG. Also, we prove that recognizing (Helly-)$B_k$-EPG graphs is in NP, for every fixed $k$. Finally, we show that recognizing Helly-$B_1$-EPG graphs is NP-complete, and it remains NP-complete even when restricted to 2-apex and 3-degenerate graphs.

Now, let $r$ be a positive integer and let $K_{2r}^-$ be the cocktail-party graph, i.e., a complete graph on $2r$ vertices with a perfect matching removed. Since $K_{2r}^-$ has $2^r$ maximal cliques, by Theorem 19 follows that $\frac{2^r}{4r} - 1 \leq b_H(K_{2r}^-)$. This implies that, for each $k$, the graph $K_{2(k+5)}^-$ is not a Helly-$B_k$-EPG graph.

Therefore, as Pergel and Rzążewski (2017) showed that every cocktail-party graph is in $B_2$-EPG, we conclude the following.

**Lemma 36.** *Helly-$B_k$-EPG $\subsetneq$ $B_k$-EPG for each $k > 0$.*

The previous lemma suggests asking about the complexity of recognizing Helly-$B_k$-EPG graphs for each $k > 1$. Also, it seems interesting to present characterizations for Helly-$B_k$-EPG representations similar to Lemma 6 (especially for $k = 2$) as well as considering the $h$-Helly-$B_k$ EPG graphs. Regarding L-shaped graphs, it also seems interesting to analyse the classes Helly-[⌞, ⌜] and Helly-[⌞, ⌜, ⌝] (recall Thereom 14).

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# Chapter 4

# The Helly and Strong Helly numbers for $B_k$-EPG and $B_k$-VPG graphs

*Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems.*

Évariste Galois

In this chapter, we investigate two parameters in EPG and VPG graph classes. The parameters that will be studied are namely the Helly number and the strong Helly number. The parameter strong Helly number generalizes the parameter Helly number. Thus, by definition, the Helly number is a natural lower bound for the strong Helly number in any family of sets studied. In this chapter, we solve the problem of determining both the Helly and strong Helly numbers, for $B_k$-EPG, and $B_k$-VPG graphs, for each value $k$.

## 4.1 Introduction

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. Such a class was introduced in 2011 [9] and [7]. In the present chapter, we study two graph parameters of both EPG and VPG graphs, namely the Helly number and the strong Helly number.

In this chapter, we study families of subsets $\mathcal{F}$ of edge and vertex paths in a grid. For EPG graphs, the Helly number of $B_0$-families is well known and is equal to 2, since $B_0$-EPG graphs coincide with interval graphs. It is also simple to conclude that the strong Helly number of $B_0$-EPG graphs are also equal to 2. For $k = 1$, we prove that both the Helly number and the strong Helly number of the class of $B_1$-families are equal to 3. For the class of $B_2$-families, we prove that these two parameters are equal to 4. The Helly and strong Helly number for $B_3$-families equal 8, and finally, these parameters are unbounded for $k \geq 4$.

As for VPG graphs, it is simple to verify that the Helly number of $B_0$-VPG graphs equals 2, and we prove that $B_1$-VPG graphs have Helly number 4, $B_2$-VPG graphs have Helly number 6, $B_3$-VPG graphs have Helly number 12, while the Helly number for $B_4$-VPG graphs is again unbounded.

Finally, the strong Helly number equals the Helly number of $B_k$-EPG graphs, for each $k$. Similarly, for $B_k$-VPG graphs.

Following, we present all the results previously mentioned.

## 4.2 Manuscript on the Helly and Strong Helly numbers for $B_k$-EPG and $B_k$-VPG graphs

# HELLY AND STRONG HELLY NUMBERS OF $B_K$-EPG AND $B_K$-VPG GRAPHS

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**Abstract**

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. A path $P$ is a $B_k$-path if it contains at most $k$ bends. $B_k$-EPG and $B_k$-VPG graphs are the intersection graphs of $B_k$-paths on the orthogonal grid, considering the intersection of edges and vertices, respectively. A family $\mathcal{F}$ is $h$-Helly when every $h$-intersecting subfamily $\mathcal{F}'$ of it satisfies $core(\mathcal{F}') \neq \emptyset$. If for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are $h$ subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be strong $h$-Helly. The Helly number of the family $\mathcal{F}$ is the least integer $h$, such that

$\mathcal{F}$ is $h$-Helly. Similarly, the strong Helly number of $\mathcal{F}$ is the least $h$, for which $\mathcal{F}$ is strong $h$-Helly. In this paper, we solve the problem of determining both the Helly and strong Helly numbers, for $B_k$-EPG, and $B_k$-VPG graphs, for each value $k$.

**Keywords:** EPG, VPG, path, grid, bend, Helly number, strong Helly.

**2010 Mathematics Subject Classification:** 05C62 - Graph representations.

## 1. Introduction

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs. Such a class was introduced in 2011 [1] and [2]. In the present paper, we study two graph parameters of both EPG and VPG graphs, namely the Helly number and the strong Helly number.

Let $\mathcal{F}$ be a family of subsets of some universal set $U$, and $h$ an integer $\geq 1$. Say that $\mathcal{F}$ is *$h$-intersecting* when every group of $h$ sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$ is the intersection of all sets of $\mathcal{F}$, denoted $core(\mathcal{F})$.

The family $\mathcal{F}$ is *$h$-Helly* when every $h$-intersecting subfamily $\mathcal{F}'$ of it satisfies $core(\mathcal{F}') \neq \emptyset$, see e.g. [4]. On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are $h$ subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong $h$-Helly*. Clearly, if $\mathcal{F}$ is $h$-Helly then it is $h'$-Helly, for $h' \geq h$. Similarly, if $\mathcal{F}$ is strong $h$-Helly then it is strong $h'$-Helly, for $h' \geq h$.

Finally, the *Helly number* of the family $\mathcal{F}$ is the least integer $h$, such that $\mathcal{F}$ is $h$-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least $h$, for which $\mathcal{F}$ is strong $h$-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number.

A *class* $\mathcal{C}$ of families $\mathcal{F}$ of subsets of some universal set $U$ is a subcollection of the families $\mathcal{F}$ of $U$. Say that $\mathcal{C}$ is a *hereditary* class when it closed under inclusion. The *Helly number* of a class $\mathcal{C}$ of families $\mathcal{F}$ of subsets is the largest Helly number among the families $\mathcal{F}$. Similarly, the *strong Helly number* of a class $\mathcal{C}$ is the largest strong Helly number of the families of $\mathcal{C}$.

If $\mathcal{F}$ is a family of subsets and $\mathcal{C}$ a class of families, denote by $H(\mathcal{F})$ and $H(\mathcal{C})$, the Helly numbers of $\mathcal{F}$ and $\mathcal{C}$, respectively, while $sH(\mathcal{F})$ and $sH(\mathcal{C})$ represent the strong Helly numbers of $\mathcal{F}$ and $\mathcal{C}$.

In this work, we study families of subsets $\mathcal{F}$ of edge and vertex paths in a grid. In the context of edge paths, a path consists of a sequence of consecutive edges in the orthogonal grid. We call a collection of such paths an *EPG representation*, i.e., a collection of paths that represent a graph via its intersection graph (considering edge intersections). *EPG graphs* are the class of graphs that admit an EPG representation. Similarly, for vertex paths, a path consists of a sequence of consecutive vertices of the orthogonal grid and a collection of these paths form a *VPG representation* and correspond to a *VPG graph.*

Each edge has an associated direction in the grid, which can be either horizontal or vertical. A *bend* in a path is a pair of consecutive edges that have different directions. A *segment* of a path is a sequence of consecutive edges of the path, with no bends. Say that a path $P_i$ is a $B_k$*-path* if it contains at most $k$ bends. Say that $\mathcal{F}$ is a $B_k$-paths family, or simply a $B_k$-family, if each path of $\mathcal{F}$ is a $B_k$-path.

In this paper, we solve the problem for determining the Helly and strong Helly numbers, for both $B_k$-EPG and $B_k$-VPG graphs, for each value $k$.

For EPG graphs, the Helly number of $B_0$-families is well known and is equal to 2, since $B_0$-EPG graphs coincide with interval graphs. It is also simple to conclude that the strong Helly number of $B_0$-EPG graphs are also equal to 2. For $k = 1$, we prove that both the Helly number and the strong Helly number of the class of $B_1$-families are equal to 3. For the class of $B_2$-families, we prove that these two parameters are equal to 4. The Helly and strong Helly number for $B_3$-families equal 8, and finally, these parameters are unbounded for $k \geq 4$.

As for VPG graphs, it is simple to verify that the Helly number of $B_0$-VPG graphs equals 2, and we prove that $B_1$-VPG have Helly number 4, $B_2$-VPG graphs have Helly number 6, $B_3$-VPG has Helly number 12, while the Helly number for $B_4$-VPG graphs is again unbounded.

Finally, the strong Helly number equals the Helly number of $B_k$-EPG graphs, for each $k$. Similarly, for $B_k$-VPG graphs.

As for existing results, Golumbic, Lipshteyn, and Stern [9] have already shown that the strong Helly number for $B_1$-EPG graphs equal 3, and for $B_1$-VPG graphs is equal to 4. employing a different proof technique. See [11], Theorem 11.13, below:

**Theorem 1.** *[11] Let $P$ be a collection of single bend paths on a grid. If every two paths in $P$ share at least one grid-edge, then $P$ has strong Helly number 3. Otherwise, $P$ has strong Helly number 4.*

No other results concerning the strong Helly number, or no results for the Helly number of $B_k$-EPG graphs seem to have been reported in the literature. As for other classes, Golumbic and Jamison have determined the strong Helly number of the intersection of edge paths of a tree [8]. Finally, Asinowski, Cohen, Golumbic, Limouzy, Lipshteyn, and Stern have reported that the strong Helly number of $B_0$-VPG graphs equals two [1].

Deciding whether a given hypergraph is $k$-Helly can be done in polynomial time for fixed $k$, employing the characterization by Berge and Duchet [3]. For arbitrary $k$, the problem is co-NP-complete [7]. For the corresponding problems for strong $k$-Helly see [6, 7].

The paper is organized as follows. Section 2 contains some preliminary propositions and further notation. Section 3 describes the results for the Helly number of $B_k$-EPG graphs, while Section 4 contains the results of this parameter for $B_k$-VPG graphs. The strong Helly number is considered in Section 5. Final remarks are presented in the last section.

## 2. Preliminaries

The following theorem characterizes $h$-Helly families of subsets.

**Theorem 2.** *([3]): A family $\mathcal{F}$ of subsets of the universal set $U$ is $h$-Helly if and only if for every subset $U' \subseteq U$, $|U'| = h + 1$, the subfamily $\mathcal{F}'$ of $\mathcal{F}$, formed by the subsets containing at least $h$ of the $h + 1$ elements of $U'$, has a non-empty core.*

The next theorem is central to our results.

**Theorem 3.** *Let $\mathcal{C}$ be a hereditary class of families $\mathcal{F}$ of subsets of the universal set $U$, whose Helly number $H(\mathcal{C})$ equals $h$. Then there exists a family $\mathcal{F}' \in \mathcal{C}$ with exactly $h$ subsets, satisfying the following condition:*

*For each subset $P_i \in \mathcal{F}'$, there is exactly one distinct element $u_i \in U$, such that*

$$u_i \notin P_i,$$

*but $u_i$ is contained in all subsets*

$$P_j \in \mathcal{F}' \setminus P_i.$$

Proof: Let $\mathcal{C}$ be a class of families $\mathcal{F}$ of subsets $P$, each subset formed by elements $u \in U$, such that the Helly number $H(\mathcal{C})$ equals $h$. Then each family $\mathcal{F} \in \mathcal{C}$ satisfies $H(\mathcal{F}) \leq h$. Consider a family $\mathcal{F}' \in \mathcal{C}$ whose Helly number is exactly $h$, and containing exactly $h$ subsets. Such a family must exist since $\mathcal{C}$ is hereditary. Since $H(\mathcal{F}') = h$, $\mathcal{F}'$ is $h$-intersecting, and therefore $(h-1)$-intersecting. Furthermore, $\mathcal{F}'$ is not $(h-1)$-Helly. Applying Theorem 2, we conclude that there are $h$ elements $U' = \{u_1, \dots, u_h\} \subset U$, such that each set of $\mathcal{F}'$ contains at least $h-1$ elements of $U'$. Since $H(\mathcal{F}') > h-1$, $core(\mathcal{F}') = \emptyset$ and therefore there is no common element among the sets of $\mathcal{F}'$. In particular, since each set $P_i \in \mathcal{F}'$ contains at least $h-1$ elements of $U'$, and $core(\mathcal{F}') = \emptyset$, we can choose $h$ subsets $P_i$, in which each of them misses a distinct element $u_i \in U'$. Then for each subset $P_i \in \mathcal{F}$, there exists some element $u_i \notin P_i$, but $u_i \in P_j$, for all $P_j \in \mathcal{F}'$, $j \neq i$. □

Let $\mathcal{F}'$ be as in the previous theorem. It is simple to conclude that the removal of any subset from $\mathcal{F}'$ makes it an $(h-1)$-Helly family. Therefore we call $\mathcal{F}'$ a *minimal non-$(h-1)$-Helly family.* Moreover, the element $u_i \notin P_i$, contained in all subsets $P_j \in \mathcal{F}' \setminus P_i$, except $P_i$, is the *$h$-non-representative* of $P_i$.

We can apply this notion of minimal families of subsets for the $B_k$-EPG and $B_k$-VPG representations. Recall that $B_k$-EPG and $B_k$-VPG graphs are hereditary classes.

## 3. The Helly Number of $B_k$-EPG Graphs

In this section, we determine the Helly number of the classes of $B_1$-EPG, $B_2$-EPG and $B_3$-EPG graphs, and show that for $B_k$-EPG graphs, $k \geq 4$, the Helly number is unbounded. We prove the following result.

**Theorem 4.** *The Helly number of $B_k$-EPG graphs satisfy:*

(i) $H(B_1\text{-}EPG) = 3$
(ii) $H(B_2\text{-}EPG) = 4$
(iii) $H(B_3\text{-}EPG) = 8$
(iv) *$H(B_k$-EPG) is unbounded, for $k \geq 4$.*

The proof consists in determining tight lower and upper bounds, as shown in the next two subsections.

### 3.1. Lower Bounds

We present lower bounds for the Helly number, as a function of the number $k$ of bends.

**Claim 5.** *The following are lower bounds for $B_k$-EPG graphs.*

(i) $H(B_1\text{-}EPG) \geq 3$
(ii) $H(B_2\text{-}EPG) \geq 4$
(iii) $H(B_3\text{-}EPG) \geq 8$
(iv) *$H(B_k$-EPG) is unbounded for $k \geq 4$.*

***Proof.*** For each value of $k$, we exhibit a $B_k$-family of edge paths whose Helly number is the corresponding stated value. We refer to the pair of coordinates of grid points, to describe the paths.

For $k = 1$, let $\mathcal{F}$ be a family of three 1-bend paths that pairwise intersect but which have no common edge, as depicted in Figure $1(a)$. Then $\mathcal{F}$ is a 2-intersecting $B_1$-EPG family of three paths, having an empty core. Furthermore,

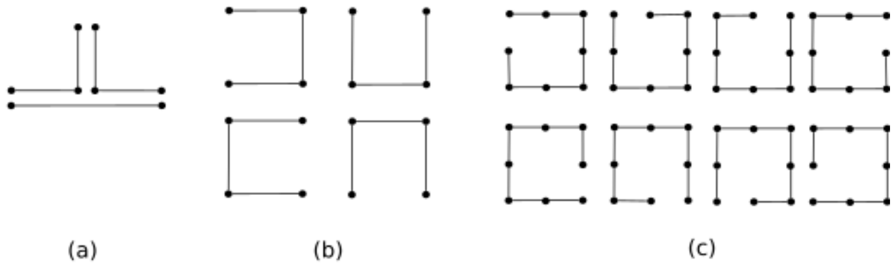


Figure 1. Minimal non-Helly sub-families for the $B_1$, $B_2$ and $B_3$ -families.

removing any of the paths from $\mathcal{F}$ makes its core become non-empty. Therefore $\mathcal{F}$ is a minimal non-2-Helly family and $H(B_1\text{-EPG}) \geq 3$.

Let $S$ be the 4-cycle formed by the four edge segments, with bends at the grid points $(0,0),(0,2),(2,2),(2,0)$, respectively. For $k=2$, consider $\mathcal{F}$ to be the family of four 2-bend paths formed when we remove exactly one of the two-edge segments that form the 4-cycle, as depicted in Figure 1($b$). It follows that $\mathcal{F}$ is 3-intersecting and there is no common edge to all four paths. Hence $H(B_2$-EPG)$\geq 4$.

For $k=3$, consider again the same cycle $S$ as above. Note that $S$ contains 8 grid edges. Let $\mathcal{F}$ consist of the 8 paths $P_i$, $1 \leq i \leq 8$, obtained by removing from $S$, exactly one of these distinct 8 edges, as depicted in Figure 1($c$). Consequently, $\mathcal{F}$ is 7-intersecting, but $core(\mathcal{F}) = \emptyset$. Therefore, $H(B_3$-EPG)$\geq 8$.

Finally, for $k=4$, let $\mathcal{F}$ be the family of $n$ paths $P_i$, described as follows:

- $P_1$ is formed by the segments connecting:
  $(0,0),(0,1),(1,1),(1,0),(n,0)$;

- for $2 \leq i \leq n-1$, $P_i$ contains the segments connecting:
  $(0,0),(0,i-1),(i-1,1),(i,1),(i,0),(n,0)$;

- $P_n$ is formed by the segments connecting:
  $(0,0),(n-1,0),(n-1,1),(n-1,0)$.

Observe that $\mathcal{F}$ is $(n-1)$-intersecting, while $core(\mathcal{F}) = \emptyset$ (see Figure 2). Therefore $H(B_4$-EPG) is unbounded. Clearly the same holds for $k > 4$. □

Next, we consider upper bounds for the Helly number $B_k$-EPG graphs.

## 3.2. Upper Bounds

In order to obtain tight upper bounds for the Helly number, in terms of the number of bends, we introduce below more notation and lemmas.

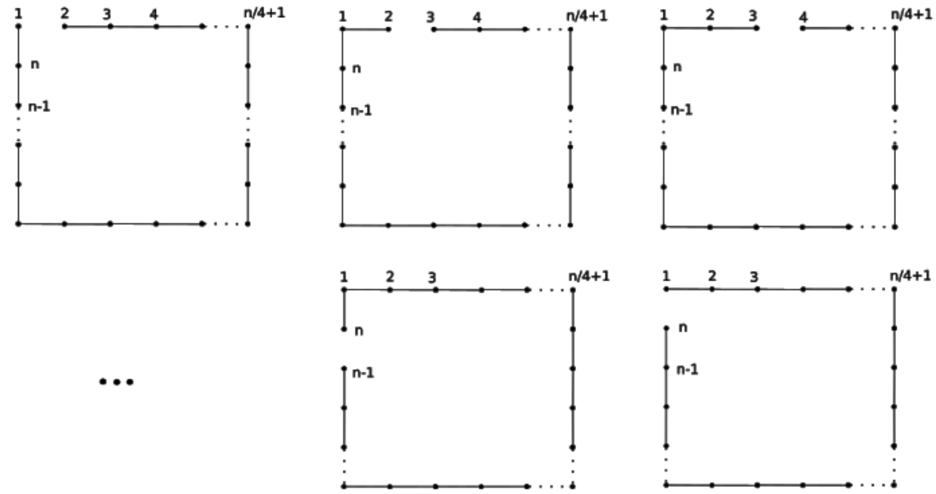


Figure 2. $B_4$ has an unlimited Helly number.

Say that a set of edges of a grid is *co-linear* if all edges of the set belong to the same line of the grid, horizontal or vertical. The set of edges is called *parallel* if all its edges lie on parallel lines of the grid, but no two of them are co-linear.

**Lemma 6.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid containing three co-linear non-representative edges. Then $\mathcal{F}$ must contain paths with at least four bends.*

***Proof.*** Let $u_i$ be the middle one of the three co-linear non-representative edges. It corresponds to the path $P_i$ of $\mathcal{F}$, not containing $u_i$. Then $P_i$ must go through the other two non-representative edges, but it cannot include the middle edge. Therefore path $P_i$ must leave the common line of the grid, containing those three representatives edges, and return to that same line, thus requiring at least four bends. □

**Lemma 7.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid, containing three parallel edges, and having Helly number $H(\mathcal{F}) \geq 4$. Then $\mathcal{F}$ must contain paths with at least four bends.*

***Proof.*** Since $H(\mathcal{F}) \geq 4$ and $\mathcal{F}$ is a minimal $(h-1)$-family, it follows that $\mathcal{F}$ must contain at least four paths, $P_1, P_2, P_3, P_4$. Without loss of generality, let $u_1, u_2, u_3$ be non-representative edges which are parallel and correspond to the paths $P_1$, $P_2$ and $P_3$ respectively. Then $P_4$ must go through all the three parallel non-representative edges $u_1, u_2, u_3$, thus requiring at least four bends. □

**Lemma 8.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths on a grid with Helly number $H(\mathcal{F}) \geq 4$. If $\mathcal{F}$ contains three non-representative edges that lie on a common $B_1$-subpath $P_i$, then $\mathcal{F}$ must have some path with at least three bends.*

***Proof.*** Since $\mathcal{F}$ is a minimal $(h-1)$-family having Helly number $\geq 4$, it contains at least four paths. Without loss of generality, let $u_1, u_2, u_3$ be the three non-representative edges contained in $P_4$ and such that $u_2$ lies between $u_1$ and $u_3$ in $P_4$. Then path $P_2$ must contain $u_1$ and $u_3$, but avoid $u_2$, thus requiring at least three bends. □

The following are tight upper bounds for the Helly numbers of $B_k$-EPG paths, for $k = 1, 2, 3$.

**Claim 9.** $H(B_1\text{-}EPG) \leq 3$.

***Proof.*** Assume by contradiction that the Helly number of $B_1$-EPG paths is $h > 3$. In this case, consider a minimal non-$(h-1)$-Helly family of $\mathcal{F}$ of $B_1$-EPG paths. Then $\mathcal{F}$ contains at least $h$ paths. Any path $P_1 \in \mathcal{F}$ must contain $h-1$ non-representative edges corresponding to the $h-1$ distinct paths of $\mathcal{F}$ other than $P_1$. Since $h-1 \geq 3$, $P_1$ contains at least three distinct non-representative edges $u_2, u_3, u_4 \in P_i$, with $u_3$ lying between $u_2$ and $u_4$ in the path.

If $u_2$, $u_3$ and $u_4$ are co-linear then by Lemma 6 $P_3 \in \mathcal{F}$ must contain at least four bends. Otherwise, the edges must lie on $P_1$ which has a single bend. Thus, it follows from Lemma 8 that $P_3$ has three bends. In any situation, a contradiction arises, implying that $H(\mathcal{F}) \leq 3$. □

**Claim 10.** $H(B_2\text{-}EPG) \leq 4$.

***Proof.*** Assume by contradiction that the Helly number of $B_2$-EPG families of paths is $h > 4$. Consider a minimal non-$(h-1)$-Helly family $\mathcal{F}$ of $B_2$-EPG paths. The family $\mathcal{F}$ must contain at least $h \geq 5$ distinct paths, each of them corresponding to a distinct non-representative edge. Choose arbitrarily 5 of these non-representative edges.

By Lemmas 6 and 7 any three of these chosen edges can neither be co-linear nor parallel. Therefore, at least one of the five chosen non-representative edges must be in a different direction from the majority of the chosen edges. Call the direction of this edge vertical and the direction of the majority of the chosen edges horizontal. Consider a path $P_1$ from the family $\mathcal{F}$ that goes through this vertical edge. The path $P_1$ contains at least four of the chosen non-representative edges, at least one of which is vertical. Since $P_1$ has at most two bends, then it must have at most three segments. Since we have three segments and four non-representative edges which $P_1$ must contain, by the pigeon hole principle, one of these segments must have two non-representative edges. If this pair of edges are in a horizontal segment of $P_1$, then such pair of edges, along with the vertical edge are in two consecutive path segments, forming a $B_1$-subpath in $\mathcal{F}$. Then Lemma 8 implies that some path of $\mathcal{F}$ must have at least three bends. Otherwise, the two edges are vertical. But the others must be horizontal, and again we have at least three edges in a pair of consecutive segments forming a subpath in $\mathcal{F}$ having one bend. Again, Lemma 8 implies that some path has at least three bends. □

**Claim 11.** *$H(B_3$-EPG$) \leq 8$.*

***Proof.*** Assume by contradiction that the Helly number of $B_3$-EPG paths is $h > 8$. In this case, consider a minimal non-$(h-1)$-Helly family $\mathcal{F}$ of $B_3$-EPG paths. Then $\mathcal{F}$ contains at least $h$ distinct non-representative edges, corresponding to $h$ distinct paths. By Lemma 7, since we can have at most three bends in any path, then these $h$ non-representative edges must lie in at most two vertical and two horizontal lines of the grid. Therefore one of these four possible lines must contain at least three distinct non-representative edges. By Lemma 6, that would imply the existence of a path with four bends. □

This completes the proof of Theorem 4.

## 4. Helly number of $B_k$-VPG Graphs

In this section, we determine the Helly number of $B_k$-VPG graphs. We prove the following results.

**Theorem 12.** *The Helly numbers for $B_k$-VPG graphs satisfy:*

1. *$H(B_1$-VPG$) = 4$*
2. *$H(B_2$-VPG$) = 6$*
3. *$H(B_3$-VPG$) = 12$*
4. *$H(B_4$-VPG$)$ is unbounded.*

Again, we prove the theorem by showing tight lower and upper bounds.

### 4.1. Lower Bounds

We start by describing some sets of paths that achieve our lower bounds. Figure 3 shows a set of 4 $B_1$-paths of a graph $G$, in a $2 \times 2$ grid, such that each path covers three vertices of the grid, and avoids exactly one of the vertices.

Figure 4 shows a set of 6 $B_2$-paths of a graph $G$, in a $2 \times 3$ grid, such that each path covers five vertices of the grid, and avoids exactly one.

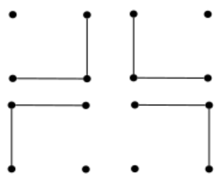

Figure 3. Lower bound for $B_1$-VPG graphs

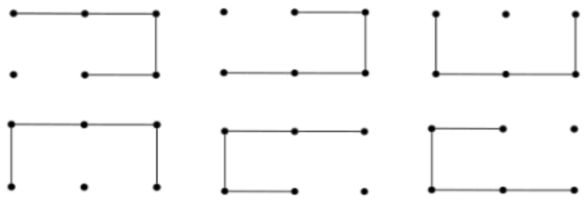

Figure 4. Lower bound for $B_2$-$VPG$ graphs

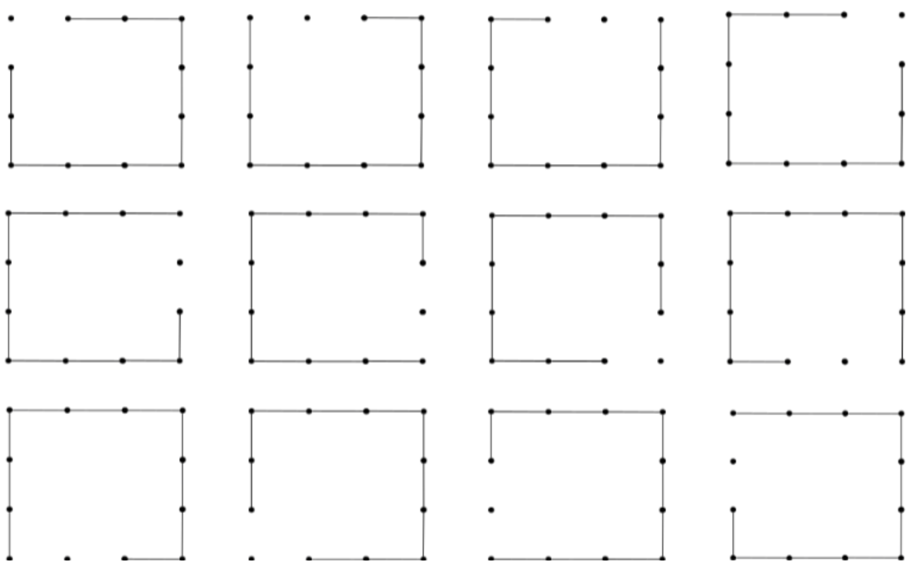

Figure 5. Lower bound for $B_3$-VPG graphs

Figure 5 shows 12 $B_3$-paths of a graph $G$, in a grid, of perimeter 12, such that each path covers 11 vertices of the grid, avoiding one of them.

Figure 6 shows a set of $n$ $B_4$-paths of a $n$-vertex graph $G$, in a grid having perimeter $n$, such that each path covers $n-1$ vertices of $G$, avoiding one of them.

Applying Theorem 3, we can then conclude that the number of vertices of each of the above-described graphs is lower bound for the corresponding class.

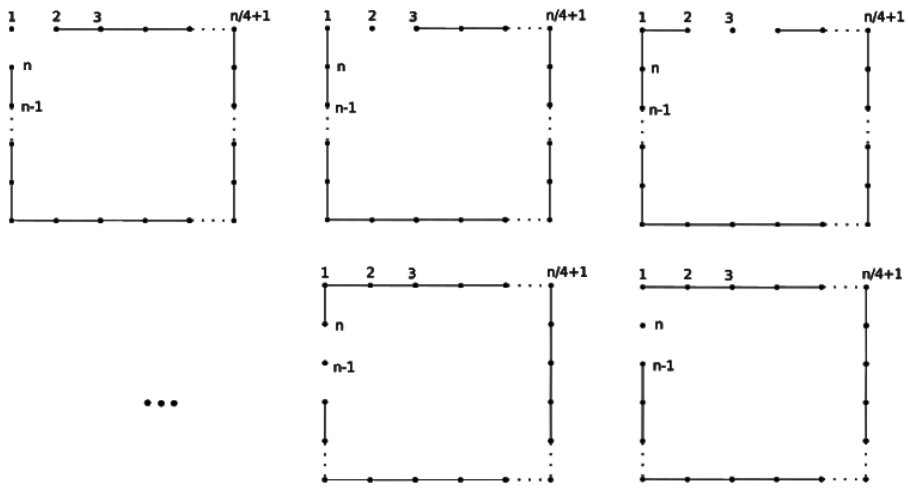


Figure 6. Lower bound for $B_4$-VPG graphs

Then, we can claim the following bounds.

**Claim 13.** *The following are lower bounds for the Helly numbers of $B_k$-VPG graphs.*

1. *$H(B_1$-VPG$) \geq 4$*
2. *$H(B_2$-VPG$) \geq 6$*
3. *$H(B_3$-VPG$) \geq 12$*
4. *$H(B_4$-VPG$)$ is unbounded.*

## 4.2. Upper Bounds

Next, we provide upper bounds for the Helly number of $B_k$-VPG graphs. The following lemmas are employed.

**Lemma 14.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths, for some $h$, containing $k \in \{3, 4, 5\}$ distinct co-linear non-representative points of the grid. Then $\mathcal{F}$ contains a path having at least $k-1$ bends.*

***Proof.*** For $k \in \{3, 5\}$, the path avoiding the middle point has at least $k-1$ bends, while for $k = 4$, the path avoiding one of the middle points also has this same property. □

**Lemma 15.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of paths, on a grid containing $k < h$ distinct pairwise non-co-linear non-representative points. Then $\mathcal{F}$ must contain a path with at least $k-1$ bends.*

***Proof.*** Since $k < h$, $\mathcal{F}$ must contain a path that visits all such $k$ pairwise non-co-linear points. Such a path requires at least one bend, between two consecutive non-co-linear points. Therefore $\mathcal{F}$ contains a path with at least $k-1$ bends. $\square$

We also employ some additional concepts and notation, below described.

Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of $B_{k-1}$-paths on a grid $Q$. By Theorem 3, we can choose $h$ paths $P_i \in \mathcal{F}$, each of them associated to a distinct non-representative grid point $p_i$, such that $P_i$ avoids $p_i$, but contains all the other $h-1$ distinct non-representative points $p_j \in P_J$, for each $j \neq i$. Denote by $P_N$, $|P_N| = h$, the subset of grid points of $Q$, restricted to the chosen set of distinct non-representative points $p_i$. By Lemmas 14 and 15, the grid points of $P_N$ are contained in at most $k$ columns (lines), and each column (line) contains at most $k$ points of $P_N$. Consequently, the cardinalities of the points of $P_N$, contained in the columns (lines) of $Q$, form a partition of the integer $h$, into at most $k$ parts, such that each part is at most $k$. Call such a partition as a *feasible partition of $h$, relative to $P_N$*. Therefore, each non-representative point $p_i \in P_N$ contributes with one unit to some part of the partition, which is then referred to, as the part of the partition *corresponding* to $p_i$.

The following lemma describes sufficient conditions for an integer $h$ to be an upper bound for the Helly number.

**Lemma 16.** *Let $\mathcal{F}$ be a minimal non-$(h-1)$-Helly family of $B_{k-1}$-paths on a grid $Q$, and $P_N$ the set of non-representative points of $Q$. Let $k, h$ be integers, $1 \leq k \leq 3$ and $k < h$. The following conditions imply $H(B_k\text{-}VPG) \leq h$*

*(i) there is no feasible partition of $h+1$, relative to $P_N$, or*

*(ii) for any possible feasible partition, and for any arrangement of the grid points of $P_N$ in $Q$, there is some non-representative point $p_i \in P_N$, such that no path exists in $Q$, having at most $k$ bends, containing all points of $P_N$, except $p_i$.*

*Proof*: The proof of (i) follows from Lemmas 14 and 15, while the proof of (ii) is a consequence of Theorem 3. $\square$

The following are upper bounds for the Helly number of $B_k$-VPG graphs, for each $k$, $1 \leq k \leq 3$, obtained by applying Lemma 16.

**Claim 17.** *$H(B_1\text{-}VPG) \leq 4$.*

***Proof.*** There is no partition of the integer 5, into two parts, in which each part is at most 2. Consequently, the result follows from Lemma 16 (i). $\square$

**Claim 18.** *$H(B_2\text{-}VPG) \leq 6$.*

***Proof.*** Assume the contrary. Then $H(B_2\text{-VPG}) \geq 7$, let $\mathcal{F}$ be a minimal non-6-Helly family of $B_2$-paths, and $P_N$ be the set of non-representative points of $\mathcal{F}$ in $Q$. There are two possible partitions of the integer 7, in three parts, each of them of size at most 3, namely $(3,3,1)$ and $(3,2,2)$. In any of these cases, it is always possible to choose some point $p_i \in P_N$, belonging to a part of the partition of size 3, such that a path in $\mathcal{F}$ which avoids $p_i$ and covers the other six non-representative points, must contain at least three bends. Then by Lemma 16, indeed $H(B_2\text{-VPG}) \leq 6$. □

**Claim 19.** *$H(B_3$-VPG$) \leq 12$.*

***Proof.*** Assume the contrary, $H(B_3\text{-VPG}) \geq 12$. Let $\mathcal{F}$ be a minimal non-12-Helly family of $B_3$-paths, and $P_N$ be the set of non-representative points of $\mathcal{F}$ in $Q$. There are three possible partitions of the integer 13, into four parts, each of them of size at most 4, namely $(4,4,4,1)$, $(4,4,3,2)$ and $(4,3,3,3)$. In this case, choose $p_i \in P_N$ to be a non-representative point, corresponding to a part of size 4 of the partition. The path of $\mathcal{F}$, which avoids $p_i$, must cover the other 12 non-representative points. These points are located in 4 distinct columns, of cardinalities 4,4,3,1, 4,3,3,2, or 3,3,3,3, considering the three possible partitions, respectively. Such a path must contain at least four bends, a contradiction. Then by Lemma 16, $H(B_3\text{-VPG}) \leq 12$. □

From the lower and upper bounds described in the previous subsections, we obtain the results for the Helly numbers of $B_k$-VPG graphs, completing the proof of Theorem 12.

## 5. Strong Helly Number

In this section, we first consider determining the strong Helly number of $B_k$-EPG graphs.

We start by describing a theorem similar to Theorem 3.

**Theorem 20.** *Let $\mathcal{C}$ be a hereditary class of families $\mathcal{F}$ of subsets of the universal set $U$, whose strong Helly number $sH(\mathcal{C})$ equals $h$. Then there exists a family $\mathcal{F}' \in \mathcal{C}$ with exactly $h$ subsets satisfying the following condition:*

*For each subset $P_i \in \mathcal{F}'$, there is exactly one distinct element $u_i \in U$, such that*

$$u_i \notin P_i,$$

*but $u_i$ is contained in all subsets*

$$P_j \in \mathcal{F}' \setminus P_i.$$

| $k$ | $B_k$-EPG | $B_k$-VPG |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 3 | 4 |
| 2 | 4 | 6 |
| 3 | 8 | 12 |
| $\geq 4$ | unbounded | unbounded |

Table 1. Helly and Strong Helly Numbers for $B_k$-EPG and $B_k$-VPG Graphs

Proof: The strong Helly number of $\mathcal{C}$ is $h$ and not $h-1$, so that there must exist some family $\mathcal{F} \in \mathcal{C}$ whose strong Helly number is exactly $h$, i.e., $\mathcal{F}$ contains $h$ subsets $P_i$ whose intersection equals core$(\mathcal{F}')$ but is such that no $h-1$ of its subsets have the same intersection. In particular, let $\mathcal{F}'$ be the family containing exactly the $h$ subsets $P_i$ described above. Such a family must exist, since $\mathcal{C}$ is hereditary. Then each $P_i$ does not contain at least one element $u_i$ in the intersection of the remaining $h-1$ subsets $P_j$, $j \neq i$, since the intersection of these $h-1$ subsets must not be equal to the core$(\mathcal{F}')$. □

Again, if we consider the family $\mathcal{F}'$ described in the theorem above it is simple to conclude that the removal of any subset from $\mathcal{F}'$ turns it $(h-1)$-strong Helly. Then call $\mathcal{F}'$ a *minimal* non-$(h-1)$-strong Helly family. Moreover, the element $u_i \notin P_i$, contained in all subsets $P_j \in \mathcal{F}' \backslash P_i$, except $P_i$, is the *h non-representative* of $P_i$.

As before, we employ the above minimal families of subsets, applied to paths in a grid.

We prove that the strong Helly number of $B_k$-EPG graphs coincide with the Helly number, for each corresponding value of $k$. Similarly, for $B_k$-VPG graphs. For $k = 0$, it is simple to show that if a set of intervals $\mathcal{I}$ in a line pairwise intersect, then there exist two intervals of $\mathcal{I}$, whose intersection equals the intersection of all intervals of $\mathcal{I}$. Consequently, the $k$-strong Helly number of $B_0$-EPG graphs equals 2. Similarly, for $B_0$-VPG graphs. Recall that the strong Helly number is at least equal to the Helly number of a family so that the lower bounds presented in Claim 5 also hold for the strong Helly number. The proofs for the strong Helly numbers for $k \geq 1$ are similar to those described in Section 3.

## 6. Concluding Remarks

We have determined the Helly number and strong Helly number of $B_k$-EPG graphs and $B_k$-VPG graphs, for $k \geq 0$.

Table 1 summarizes the results obtained.

We leave two questions to be investigated concerning the presented results.

1. Given a *specific* EPG or VPG graph, the question is to formulate an algorithm to determine its Helly and strong Helly numbers. See [5], for instance, for such algorithms, applied to general graphs.

2. The values of the Helly and strong Helly numbers, which were determined in this paper, coincided in all cases. Clearly, in general, this is not the case. We leave as an open question, to find the conditions for such equality to occur.

## Acknowledgements

The authors thank Martin Golumbic for helpful suggestions.

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# Chapter 5

# Relationship among $B_1$-EPG, EPT and VPT graph classes

*What we know is a drop, what we don't know is an ocean.*

Sir Isaac Newton

This chapter presents as the main result the proof that every Chordal $B_1$-EPG graph is simultaneously in the VPT and EPT graph classes. In particular, we describe structures that belong to $B_1$-EPG but do not support a Helly-$B_1$-EPG representation and thus we define some sets of subgraphs that delimit Helly subfamilies. Besides, this chapter also presents features of some non-trivial graph families that are properly contained in Helly-$B_1$ EPG, namely these families are composed by Bipartite, Blocks, Cactus, and Line of Bipartite graphs.

## 5.1 Introduction

Models based on paths intersection may consider intersections by vertices or intersections by edges. Cases where the paths are hosted on a tree have appeared in [36, 39, 41], among others. Representations using paths on a grid were considered later, see [42, 46, 47].

A pertinent question in the context of path intersection graphs is as follows: given two classes of path intersection graphs, the first whose host is a tree and the second whose host is a grid, is there an intersection or containment relationship among these classes? What do we know about it?

In the present chapter we will explore $B_1$-EPG graphs, in particular Diamond-free graphs and Chordal graphs. We will work on the question about the containment relation between VPT, EPT and $B_1$-EPG graph classes.

We presented an infinite family of forbidden induced subgraphs for the class $B_1$-EPG and in particular we proved that Chordal $B_1$-EPG $\subset$ VPT $\cap$ EPT. In addition, we also propose other questions for future research.

Next, we present the manuscript where the reader can find all the results previously mentioned.

## 5.2 Manuscript on $B_1$-EPG and EPT Graphs

# ON $B_1$-EPG AND EPT GRAPHS

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**Abstract**

This research contains as a main result the proof that every Chordal $B_1$-EPG graph is simultaneously in the graph classes VPT and EPT. In addition, we describe structures that must be present in any $B_1$-EPG graph which does not admit a Helly-$B_1$-EPG representation. In particular, this paper presents some features of non-trivial families of graphs properly contained in Helly-$B_1$-EPG, namely Bipartite, Block, Cactus and Line of Bipartite graphs.

**Keywords:** Edge-intersection of paths on a grid, Edge-intersection graph of paths in a tree, Helly property, Intersection graphs, Single bend paths, Vertex-intersection graph of paths in a tree.

**2010 Mathematics Subject Classification:** 05C62 - Graph representations.

## 1. Introduction

Models based on paths intersection may consider intersections by vertices or intersections by edges. Cases where the paths are hosted on a tree appear first in the literature, see for instance [9, 10, 11]. Representations using paths on a grid were considered later, see [12, 13, 15].

Let $P$ be a family of paths on a host tree $T$. Two types of intersection graphs from the pair $< P, T >$ are defined, namely VPT and EPT graphs. The *edge intersection graph* of $P$, EPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in EPT(P) if and only if the corresponding paths in $P$ share at least one edge in T. Similarly, the *vertex intersection graph* of $P$, VPT(P), has vertices which correspond to the members of $P$, and two vertices are adjacent in VPT(P) if and only if the corresponding paths in $P$ share at least one vertex in $T$. VPT and EPT graphs are incomparable families of graphs. However, when the maximum degree of the host tree is restricted to three the family of VPT graphs coincides with the family of EPT graphs [10]. Also it is known that any Chordal EPT graph is VPT (see [19]). Recall that it was shown that Chordal graphs are the vertex intersection graphs of subtrees of a tree [8].

Edge intersection graphs of paths on a grid are called *EPG graphs*.

In [12], the authors proved that every graph is EPG, and started the study of the subclasses defined by bounding the number of times any path used in the representation can bend. Graphs admitting a representation where paths have at most $k$ changes of direction (bends) were called $B_k$-EPG. In particular, when the paths have at most one bend we have the *$B_1$-EPG graphs* or a *single bend EPG graphs*.

A pertinent question in the context of path intersection graphs is as follows: given two classes of path intersection graphs, the first whose host is a tree and the second whose host is a grid, is there an intersection or containment relationship among these classes? What do we know about it?

In the present paper we will explore $B_1$-EPG graphs, in particular diamond-free graphs and Chordal graphs. We will work on the question about the containment relation between VPT, EPT and $B_1$-EPG graph classes.

A collection of sets satisfies the *Helly property* when every pair-wise intersecting sub-collection has at least one common element. When this property is satisfied by the set of vertices (edges) of the paths used in a representation, we get a Helly representation. Helly-$B_1$-EPG graphs were studied in [5]. It is known that not every $B_1$-EPG graph admits a Helly-$B_1$-EPG representation. We are interested in determining the subgraphs that make $B_1$-EPG graphs that do not admit a Helly representation. In the present work, we describe some structures that will be present in any such subgraph, and, in addition, we present new Helly-$B_1$-EPG subclasses. Moreover, we describe new Helly-$B_1$-EPG subclasses and we give some sets of subgraphs that delimit Helly subfamilies.

## 2. Definitions and Technical Results

The *vertex set* and the *edge set* of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. Given a vertex $v \in V(G)$, $N(v)$ represents the *open neighborhood* of $v$ in $G$. For a subset $S \subseteq V(G)$, $G[S]$ is the subgraph of $G$ induced by $S$. If $\mathcal{F}$ is any family of graphs, we say that $G$ is *$\mathcal{F}$-free* if $G$ has no induced subgraph isomorphic to a member of $\mathcal{F}$. A *cycle*, denoted by $C_n$, is a sequence of distinct vertices $v_1, \ldots, v_n, v_1$ where $v_i \neq v_j$ for $i \neq j$ and $(v_i, v_i + 1) \in E(G)$, such that $n \geq 3$. A *chord* is an edge that is between two non-consecutive vertices in a sequence of vertices of a cycle. An *induced cycle* or *chordless cycle* is a cycle that has no chord, in this paper an induced cycle will simply be called a *cycle.* A graph $G$ formed by an induced cycle $H$ plus a single universal vertex $v$ connected to all vertices of $H$ is called a *wheel graph.* If the wheel has $n$ vertices, it is denoted by $n$-wheel.

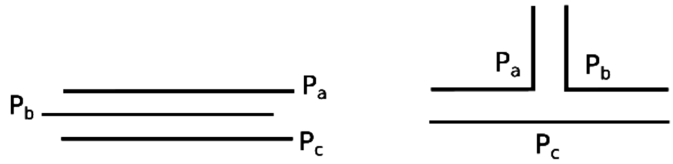
The *$k$-sun graph* $S_k$, $k \geq 3$, consists of $2k$ vertices, an independent set $X = \{x_1, \ldots, x_k\}$ and a clique $Y = \{y_1, \ldots, y_k\}$, and edge set $E_1 \cup E_2$, where $E_1 = \{(x_1, y_1); (y_1, x_2); (x_2, y_2); (y_2, x_3); \ldots, (x_k, y_k); (y_k, x_1)\}$ and $E_2 = \{(y_i, y_j) | i \neq j\}$.

A graph is a $B_k$-EPG graph if it admits an EPG representation in which each path has at most $k$ bends. When $k = 1$ we say that this is a *single bend EPG* representation or simply a $B_1$-EPG representation. A *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non adjacent vertices. Given an EPG representation of a graph $G$, we will identify each vertex $v$ of $G$ with the corresponding path $P_v$ of the grid used in the representation. Accordingly, for instance, we will say that a vertex of $G$ covers or contains some edge of the grid (meaning that the corresponding path does), or that a set of paths of the representation induces a subgraph of $G$ (meaning that the corresponding set of vertices does).

In a $B_1$-EPG representation, a clique $K$ is said to be an *edge-clique* if all the vertices of $K$ share a common edge of the grid (see Figure 1(a)). A *claw of the grid* is a set of three edges of the grid incident into the same point of the grid, which is called the *center of the claw.* The two edges of the claw that have the same direction form the *base of the claw.* If $K$ is not an edge-clique, then there exists a claw of the grid (and only one) such that the vertices of $K$ are those containing exactly two of the three edges of the claw; such a clique is called *claw-clique* [12] (see Figure 1(b)).

Notice that if three vertices induce a claw-clique, then exactly two of them turn at the center of the corresponding claw of the grid, and the third one contains the base of the claw. Furthermore, any other vertex adjacent to the three must contain two of the edges of that claw, then the following lemma holds.

**Lemma 1.** *If three vertices are together in more than one maximal clique of a*




(a) Representation of a clique as edge-clique.

(b) Representation of a clique as claw-clique.

Figure 1. Examples of clique representations.

*graph $G$, then in any $B_1$-EPG representation of $G$ the three vertices do not form a claw-clique.*

In [3] Asinowski et al. proved the following lemma for $C_4$-free graphs.

**Lemma 2.** *[3] Let $G$ be a $B_1$-EPG graph. If $G$ is $C_4$-free, then there exists a $B_1$-EPG representation of $G$ such that every maximal claw-clique $K$ is represented on a claw of the grid whose base is covered only by vertices of $K$.*

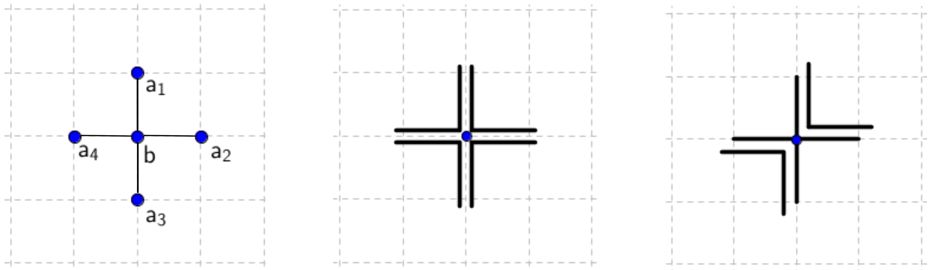
We have obtained the following similar result for diamond-free graphs. A *diamond* is a graph $G$ with vertex set $V(G) = \{a, b, c, d\}$ and edge set $E(G) = \{ab, ac, bc, bd, cd\}$.

**Lemma 3.** *Let $G$ be a $B_1$-EPG graph. If $G$ is diamond-free, then in any $B_1$-EPG representation of $G$, every maximal claw-clique $K$ is represented on a claw of the grid whose edges are covered only by vertices of $K$.*

***Proof.*** Let $K$ be a maximal clique which is a claw-clique in a given $B_1$-EPG representation of $G$. Then there exist three vertices of $K$ which induce a claw-clique $K'$ on the same claw of the grid than $K$. Assume, in order to derive a contradiction, that a vertex $v \notin K$ covers some edge of the claw. Clearly, $v$ must cover only one of such edges. Therefore $v$ and the vertices of $K'$ induce a diamond, a contradiction. ■

Let $Q$ be a grid and let $(a_1, b)$, $(a_2, b)$, $(a_3, b)$, $(a_4, b)$ be a 4-star centered at $b$ as depicted in Figure 2(a). Let $\mathcal{P} = \{P_1, \ldots, P_4\}$ be a collection of four paths each containing a different pair of edges of the 4-star. Following [12], we say that the four paths form

- a *true pie* when each one has a bend at $b$, Figure 2(b); and
- a *false pie* when exactly two of the paths bend at $b$ and they do not share an edge of the 4-star, Figure 2(c).




(a) 4-star in grid. (b) True pie. (c) False pie.

Figure 2. $B_1$-EPG representation of the induced cycle of size 4 as pies with emphasis in center $b$.

Clearly if four paths of a $B_1$-EPG representation of $G$ form a pie, then the corresponding vertices induce a 4-cycle in $G$. The following result can be easily proved. We say that a set of paths form a claw when each pair of edges of the claw is covered by some of the paths.

**Lemma 4.** *In any $B_1$-EPG representation of a graph $G$, a set of paths forming two different claws centered at the same point of the grid contains four paths forming either a true pie or a false pie. Therefore, in any $B_1$-EPG representation of a chordal graph $G$, no two maximal claw-cliques of $G$ are centered at the same point of the grid.*

**Lemma 5.** *Let $G$ be a graph whose vertex set can be partitioned into a non trivial clique $K$ and an independent set $I = \{w_1, w_2, w_3\}$, such that each vertex of $K$ is adjacent to each vertex of $I$. Then, in any $B_1$-EPG representation of $G$, at least one of the cliques $K_i = K \cup \{w_i\}$, with $1 \leq i \leq 3$, is an edge-clique.*

***Proof.*** Assume, in order to derive a contradiction, that the three cliques are claw-cliques. By Lemma 4, they have different centers, say the points $q_1, q_2, q_3$ of the grid, respectively. Since at least two paths have a bend at the center of a claw, for each $i \in \{1, 2, 3\}$, there must exist a vertex $v_i$ of $K$ such that the corresponding path $P_{v_i}$ turns at the point $q_i$ of the grid. Notice that each one of the three paths $P_{v_i}$ must contain the three grid points $q_1$, $q_2$ and $q_3$. To prove that this is not possible, we will consider, without loss of generality, two cases. First, $q_1$ is between $q_2$ and $q_3$ in $P_{v_1}$. Then, $P_{v_3}$ cannot turn at $q_3$ and contain $q_1$ and $q_2$. And second, $q_2$ is between $q_1$ and $q_3$ in $P_{v_1}$. In this case, $P_{v_2}$ cannot turn at $q_2$ and contain $q_1$ and $q_3$; thus the proof is completed. ■

Three vertices $u, v, w$ of a graph $G$ form an *asteroidal triple* (AT) of $G$ if for every pair of them there exists a path connecting the two vertices and such that the path avoids the neighborhood of the remaining vertex [4]. A graph without an asteroidal triple is called *AT-free*.

**Lemma 6** [3]. *Let $v$ be any vertex of a $B_1$-EPG graph $G$. Then $G[N(v)]$ is AT-free.*

Let $C$ be any subset of the vertices of a graph $G$. The *branch graph* $B(G|C)$, see [12], of $G$ over $C$ has a vertex set, $V(B)$, consisting of all the vertices of $G$ not in $C$ but adjacent to some member of $C$, i.e. $V(B) = N(C) - C$. Adjacency in $B(G|C)$ is defined as follows: we join two vertices $x$ and $y$ by an edge in $E(B)$ if and only if in $G$ occurs:

1. $x$ and $y$ are not adjacent;
2. $x$ and $y$ have a common neighbor $u \in C$;
3. the sets $N(x) \cap C$ and $N(y) \cap C$ are not comparable, i.e. there exist private neighbors $w, z \in C$ such that $w$ is adjacent to $x$ but not to $y$, and $z$ is adjacent to $y$ but not to $x$; we say that $x$ and $y$ are neighborhood incomparable.

We let $\chi(G)$ denote the chromatic number of $G$.

**Lemma 7** [12]. *Let $C$ be any maximal clique of a $B_1$-EPG graph $G$. Then, the branch graph $B(G|C)$ is $\{P_6, C_n$ for $n \geq 4\}$-free, and $\chi(B(G/C)) \leq 3$.*

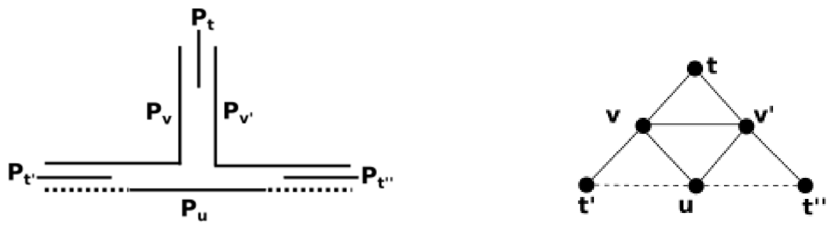
## 3. SUBCLASSES OF HELLY-$B_1$-EPG GRAPHS

In this section, we delimit some subclasses of $B_1$-EPG graphs that admit a Helly-$B_1$-EPG representation. It is known that $B_1$-EPG and Helly-$B_1$-EPG are hereditary classes, so they can be characterized by forbidden structures. In both cases, finding the list of minimal forbidden induced subgraphs are challenging open problems. Taking a step towards solving those problems, we describe a few structures at least one of which will necessarily be present in any $B_1$-EPG graph that does not admit a Helly representation. In addition, we show that the well known families of Block graphs, Cactus and Line of Bipartite graphs are totally contained in the class Helly-$B_1$-EPG.

Let $S_3, S_{3'}, S_{3''}$ and $C_4$ be the graphs depicted in Figure 4.

**Theorem 8.** *Let $G$ be a $B_1$-EPG graph. If $G$ is $\{S_3, S_{3'}, S_{3''}, C_4\}$-free then $G$ is a Helly-$B_1$-EPG graph.*

***Proof.*** If $G$ is not a Helly-$B_1$-EPG graph, then in each $B_1$-EPG representation of $G$, there is at least one clique that is represented as claw-clique and not as edge-clique. Consider any $B_1$-EPG representation of $G$ and let $K$ be a maximal clique which is represented as a claw-clique. Assume, w.l.o.g, $K$ is on a claw of the grid with base $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. Denote by $\mathcal{P}_K$ the




(a) Claw with paths. (b) Subgraph induced by paths.

Figure 3. Reconstruction of the intersection model.

set of paths corresponding to the vertices of $K$. By Lemma 2, the grid segment $[x_0, x_2] \times \{y_0\}$ is covered only by vertices of $K$.

For every $\lrcorner$-path (resp. $\llcorner$-path ) belonging to $\mathcal{P}_K$, we do the following: if the path does not intersect any path $P_t \notin \mathcal{P}_K$ on column $x_1$, then we delete its vertical segment and add the grid segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_1] \times \{y_0\}$). If after this transformation there is no more $\lrcorner$-paths (resp. $\llcorner$-paths) in $\mathcal{P}_K$, then we are done since we have obtained an edge-clique. So we may assume that every $\lrcorner$-path and every $\llcorner$-path in $\mathcal{P}_K$ intersects some path $P_t \notin \mathcal{P}_K$ on column $x_1$ (notice that we can assume is the same path $P_t$ for all the vertices).

Now, if none of the $\lrcorner$-paths belonging to $\mathcal{P}_K$ intersect a path not in $\mathcal{P}_K$ on the line $y_0$, then we can replace the horizontal part of those paths by the segment $[x_1, x_2] \times \{y_0\}$, getting an edge representation of the clique $K$. Thus, we can assume there exists at least one $\lrcorner$-path $P_v \in \mathcal{P}_K$ intersecting some path $P_{t'} \notin \mathcal{P}_K$ on line $y_0$. Analogously, there exists at least one $\llcorner$-path $P_{v'} \in \mathcal{P}_K$ intersecting some path $P_{t''} \notin K$ on line $y_0$, as depicted in Figure 3. Notice that vertex $t'$ cannot be adjacent to any of the vertices $t$, $v'$ or $t''$; and, in addition, vertex $t''$ cannot be adjacent to $t$, or $v$.

Finally, since $K$ is claw-clique, there is a path $P_u \in \mathcal{P}_K$ covering the base of the claw. Depending on the possible adjacencies between $u$ and $t'$ or $t''$, one of the graphs $S_3$, $S_{3'}$ or $S_{3''}$ is obtained.

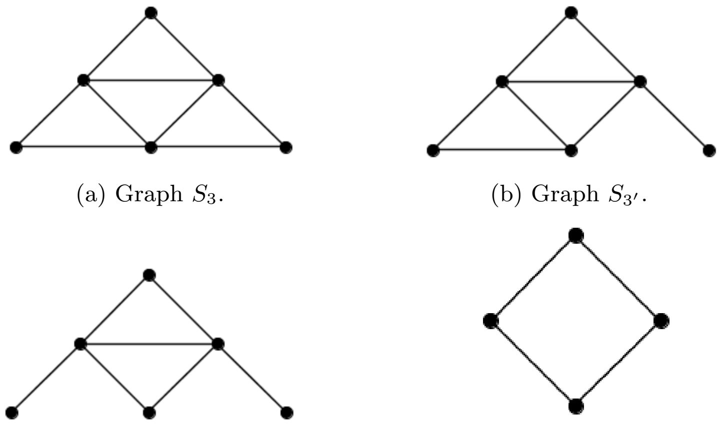
■

Notice that any bull-free graph is $\{S_3, S_{3'}, S_{3''}\}$-free, so our previous result implies Lemma 5 of [3].

Next theorem has as consequence the identification of several graph classes where the existence of a $B_1$-EPG representation ensures the existence of a Helly-$B_1$-EPG representation.

**Theorem 9.** *If $G$ is a $B_1$-EPG and diamond-free graph then $G$ is a Helly-$B_1$-EPG graph.*

***Proof.*** If $G$ is not a Helly-$B_1$-EPG graph, then in each $B_1$-EPG representation



(a) Graph $S_3$. (b) Graph $S_{3'}$.

(c) Graph $S_{3''}$. (d) Graph $C_4$.

Figure 4. Graphs on the statement of Theorem 8.

of $G$, there is at least one clique that is represented as claw-clique and no as edge-clique. Consider any $B_1$-EPG representation of $G$ and let $K$ be a maximal clique which is represented as a claw-clique. Assume, w.l.o.g, $K$ is on a claw of the grid with base $[x_0, x_2] \times \{y_0\}$ and center $C = (x_1, y_0)$. Denote by $\mathcal{P}_K$ the set of paths corresponding to the vertices of $K$. By Lemma 3, the grid segment $[x_0, x_2] \times \{y_0\}$ is covered only by vertices of $K$. For every $\lrcorner$-path (resp. $\llcorner$-path ) belonging to $\mathcal{P}_K$, we do the following: if the path does not intersect any path $P_t \notin \mathcal{P}_K$ on column $x_1$, then we delete its vertical segment and add the grid segment $[x_1, x_2] \times \{y_0\}$ (resp. $[x_0, x_1] \times \{y_0\}$). If after this transformation there is no more $\lrcorner$-paths (resp. $\llcorner$-paths) in $\mathcal{P}_K$, then we are done since we have obtained an edge-clique. So we may assume that every $\lrcorner$-path and every $\llcorner$-path in $\mathcal{P}_K$ intersects some path $P_t \notin \mathcal{P}_K$ on column $x_1$ (notice that we can assume is the same path $P_t$ for all the vertices). Since $K$ is claw-clique, there is a path $P_u \in \mathcal{P}_K$ covering the base of the claw. Thus, $G[v, v', u, t]$ induces a diamond, a contradiction. ■

An *independent set* of vertices is a set of vertices no two of which are adjacent. A graph $G$ is said to be *Bipartite* if its set of vertices can be partitioned into two distinct independent sets. There are Bipartite graphs that are not $B_1$-EPG, for instance $K_{2,5}$ and $K_{3,3}$ (see [7]). Clearly, since bipartite graphs are triangle-free, any $B_1$-EPG representation of a bipartite graph is also a Helly-$B_1$-EPG representation. A similar result (but a bit weaker) is obtained as a corollary of the previous theorem.

**Corollary 10.** *If $G$ is a Bipartite $B_1$-EPG graph then $G$ is a Helly-$B_1$-EPG graph.*

***Proof.*** The Bipartite graphs are diamond-free, thus by Theorem 9 these graphs are Helly-$B_1$-EPG graphs. ■

A *Block graph* or *Clique Tree* is a type of graph in which every biconnected component (block) is a clique.

**Corollary 11.** *Block graphs are Helly-$B_1$-EPG.*

***Proof.*** Block graphs are known to be exactly the Chordal diamond-free graphs, so by Theorem 19 of [3], all Block graphs are $B_1$-EPG. If follows from Theorem 9 that all Block graphs are Helly-$B_1$-EPG. ■

A *Cactus* (sometimes called a Cactus Tree) graph is a connected graph in which any two cycles have at most one vertex in common. Equivalently, it is a connected graph in which every edge belongs to at most one cycle, or (for nontrivial Cactus) in which every block (maximal subgraph without a cut-vertex) is an edge or a cycle. The family of graphs in which each component is a Cactus is closed under graph minor operations. This graph family may be characterized by a single forbidden minor, the diamond graph.

**Corollary 12.** *Cactus graphs are Helly-$B_1$-EPG.*

***Proof.*** In [6], it is proved that every Cactus graph is a monotonic $B_1$-EPG graph (there is a $B_1$-EPG representation where all paths are ascending in rows and columns). Thus, Cactus graphs are $B_1$-EPG graphs.

Since Cactus are diamond-free, by Theorem 9, the proof follows. ■

Given a graph $G$, its *Line graph* $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of $G$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint (i.e. "are incident") in $G$. A graph $G$ is a *Line graph of a Bipartite graph* (or simply *Line of Bipartite*) if and only if it contains no claw, no odd cycle (with more than 3 vertices), and no diamond as an induced subgraph [16].

In [17] was proved that every Line graph has a representation with at most 2 bends. We proved in the following corollary that when restricted to the Line of Bipartite we can obtain a representation Helly and one-bended.

**Corollary 13.** *Line of Bipartite graphs are Helly-$B_1$-EPG.*

***Proof.*** Line of Bipartite graphs were proved to be $B_1$-EPG in [14]. Since they are diamond-free, the proof follows from Theorem 9.

■

The diagram of Figure 5 illustrates the containment relationship between the graph classes studied so far in this work. We list in Figure 6 examples of graphs in each numbered region of the diagram. The numbers of each item below correspond to the regions of the same number in the diagram depicted in Figure 5.

(1) ($B_1$-EPG) - (Helly-$B_1$-EPG) graphs, depicted in Figure 6(a), graph $E_1$;

(2) (Line of Bipartite) - (Cactus) - (Block) - (Bipartite) graphs, depicted in Figure 6(b), graph $E_2$;

(3) (Helly-$B_1$-EPG) - (Line of Bipartite) - (Block) - (Cactus) - (Bipartite) graphs, depicted in Figure 6(c), graph $E_3$;

(4) (Block) $\cap$ (Line of Bipartite) - (Cactus) - (Bipartite), depicted in Figure 6(d), graph $E_4$;

(5) (Block) $\cap$ (Line of Bipartite) $\cap$ (Cactus) - (Bipartite), depicted in Figure 6(e), graph $E_5$;

(6) (Cactus) $\cap$ (Line of Bipartite) - (Block) - (Bipartite). This intersection is empty. Let $G$ be a graph that is Cactus and Line of Bipartite then $G$ is {claw, odd cycle, diamond}-free. But $G$ is not a Bipartite graph, then $G$ has odd cycle, absurd with the hypothesis of $G$ is Line of Bipartite;

(7) (Bipartite) $\cap$ (Line of Bipartite) - (Cactus) - (Block) graphs, depicted in Figure 6(f), graph $E_7$;

(8) (Bipartite) $\cap$ (Line of Bipartite) $\cap$ (Cactus) - (Block) graphs, depicted in Figure 6(g), graph $E_8$;

(9) (Bipartite) $\cap$ (Line of Bipartite) $\cap$ (Cactus) $\cap$ (Block) graphs, depicted in Figure 6(h), graph $E_9$;

(10) (Bipartite) $\cap$ (Cactus) $\cap$ (Block) - (Line of Bipartite) graphs, depicted in Figure 6(i), graph $E_{10}$;

(11) (Bipartite) $\cap$ (Cactus) - (Block) - (Line of Bipartite) graphs, depicted in Figure 6(j), graph $E_{11}$;

(12) (Bipartite) $\cap$ (Helly-$B_1$-EPG) - (Cactus) - (Block) - (Line of Bipartite) graphs, depicted in Figure 6(k), graph $E_{12}$;

(13) (Bipartite) - ($B_1$-EPG) graphs, depicted in Figure 6(l), graph $E_{13}$;

(14) (Block) - (Bipartite) - (Line of Bipartite) - (Cactus) graphs, depicted in Figure 6(m), graph $E_{14}$;

(15) (Block) ∩ (Cactus) - (Line of Bipartite) - (Bipartite) graphs, depicted in Figure 6(n), graph $E_{15}$;

(16) (Cactus) - (Block) - (Line of Bipartite) - (Bipartite) graphs, depicted in Figure 6(o), graph $E_{16}$, the odd cycles $C_{2n+1}, n \geq 2$;

(17) (Helly EPG) - ($B_1$-EPG) - (Bipartite) graphs, depicted in Figure 6(p), graph $E_{17}$;

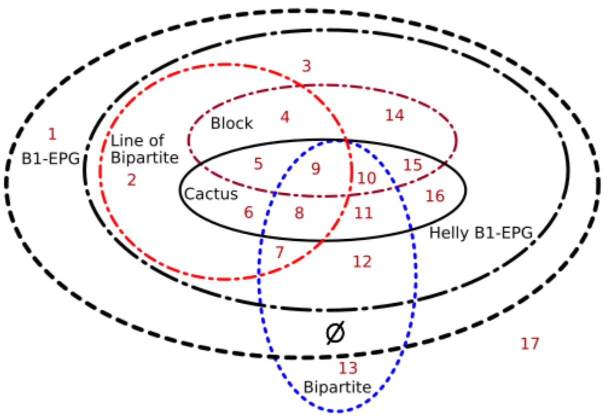
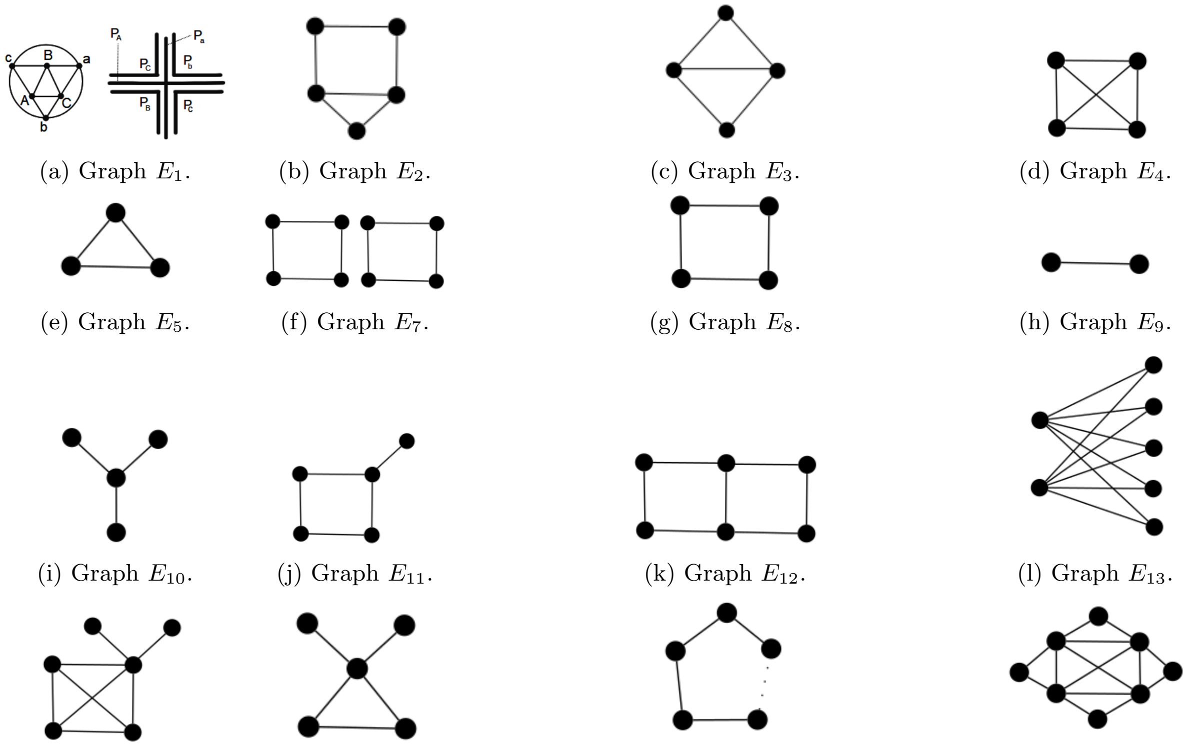



Figure 5. Diagram of some graph classes.

In the next section we explore the Chordal $B_1$-EPG graphs through of a subset of forbidden graphs and we will prove that this class is in the strict intersection of VPT and EPT graphs.

## 4. Containment relationship among Chordal $B_1$-EPG, VPT and EPT graphs

Any graph that admits a $B_1$-EPG representation whose paths do not cover all the edges of a polygon of the grid (i.e. the subjacent grid subgraph is a tree) is also an EPT graph: the same representation is both $B_1$-EPG and $EPT$. However, it is easily verifiable that the subjacent grid subgraph of any $B_1$-EPG representation of a cycle $C_n$ with $n \geq 5$ is not a tree, although $C_n$ is an EPT graph. Our long-range goal is understanding the $B_1$-EPG graphs that are also EPT graphs. When can a $B_1$-EPG representation be reorganized into an EPT representation? In this section, we answer that question for Chordal $B_1$-EPG graphs, in fact we prove that every Chordal $B_1$-EPG graph is EPT. We made several unsuccessful attempts to prove this result by considering for a graph $G$, a $B_1$-EPG representation whose paths cover all the edges of some polygon on the grid, and trying




(a) Graph $E_1$. (b) Graph $E_2$. (c) Graph $E_3$. (d) Graph $E_4$.

(e) Graph $E_5$. (f) Graph $E_7$. (g) Graph $E_8$. (h) Graph $E_9$.

(i) Graph $E_{10}$. (j) Graph $E_{11}$. (k) Graph $E_{12}$. (l) Graph $E_{13}$.

(m) Graph $E_{14}$. (n) Graph $E_{15}$. (o) Graph $E_{16}$, $C_{2n+1}, n \geq 2$. (p) Graph $E_{17}$.

Figure 6. The set of instances for the Venn Diagram on Figure 5.

to show that if none of the paths could be modified in order to avoid an edge of the polygon, then $G$ had some chordless cycle (i.e. $G$ is not chordal). The surprise was that the only way we found to demonstrate our main Theorem 23 was through $VPT$ graphs. We will prove the following theorem.

**Theorem 14.** *Chordal $B_1$-EPG $\subsetneq$ VPT.*

In Lévêque et al. [18] apud [2], VPT graphs were characterized by a family of minimal forbidden induced subgraphs, the ones depicted in Figure 7 plus the induced cycles $C_n$ for $n \geq 4$. Therefore, in order to prove that Chordal $B_1$-EPG graphs are VPT is enough to show that none of the graphs in Figure 7 is $B_1$-EPG.

First notice that in each one of the graphs $F_1, F_2, F_3, F_4$ and $F_5$ ( Figures 7(a), (b), (c), (d), (e), respectively), the neighborhood of the universal vertex (the one that is a bit bigger than the others, in the respective figures) contains an asteroidal triple. Therefore, by Lemma 6, these graphs are not $B_1$-EPG.

Now, in each one of the graphs $F_{11}, F_{12}, F_{13}, F_{14}$, $F_{15}$ and $F_{16}$ (Figures 7(k), (l), (m), (n), (o), (p), respectively), let $C$ be the maximal clique in bold. It is easy to check that, in all cases, the branch graph $B(G|C)$ contains an induced cycle $C_n$, for some $n \geq 4$, or an induced path $P_6$; thus, by Lemma 7, graphs $F_{11}, F_{12}, F_{13}, F_{14}$, $F_{15}$ and $F_{16}$ are not $B_1$-EPG.

**Observation 15.** *Let $e_\ell$, $e_m$ and $e_r$ be three distinct edges of a one-bend path $P$, and assume that $e_m$ is between $e_\ell$ and $e_r$ on $P$. If $P_\ell$ and $P_r$ are one-bend paths such that: $P_\ell$ contains $e_\ell$, $P_r$ contains $e_r$, and $P_\ell$ and $P_r$ intersect in at least one edge, then $P_\ell$ or $P_r$ contains $e_m$.*

**Observation 16.** *Let $e$ and $q$ be an edge and a point of a one-bend path $P$, respectively. If a one-bend path $P'$ contains both $e$ and $q$, then $P'$ contains the whole segment of $P$ between $q$ and $e$.*

**Lemma 17.** *Let $G$ be a graph whose vertex set can be partitioned into a clique $K = \{a, b\}$ and an independent set $I = \{x, y, z\}$, such that each vertex of $K$ is adjacent to each vertex of $I$. If in a given $B_1$-EPG representation of $G$, $P_a \cap P_y$ is between $P_a \cap P_x$ and $P_a \cap P_z$, then $\{a, b, y\}$ is an edge-clique, and $P_a \cap P_y \subset P_b$. Even more, any vertex adjacent to both $a$ and $y$, but not to $b$ (or to $b$ and $y$, but not to $a$) has to be adjacent to $x$ or to $z$.*

***Proof.*** Assume in order to obtain a contradiction that $\{a, b, y\}$ is not an edge-clique. Then, by Lemma 5, we can assume, w.l.o.g., that $\{a, b, x\}$ is an edge-clique. It implies that there is an edge $e_\ell$ of $P_a \cap P_x$ covered by $P_b$. Since every edge of $P_a \cap P_z$ is covered by $P_z$, $z$ and $b$ are adjacent, and $z$ and $y$ are non adjacent, we have by Observation 15, that every edge of $P_a \cap P_y$ is covered by $P_b$, which implies that $\{a, b, y\}$ is an edge-clique, contrary to the assumption.

Thus, $\{a, b, y\}$ is an edge-clique. By Observation 16, we have that the whole interval of $P_a$ between $P_a \cap P_x$ and $P_a \cap P_z$ is contained in $P_b$, and so, in particular, $P_a \cap P_y \subset P_b$. Observe that this implies that if $q$ is an end point of the interval $P_a \cap P_y$, and $e$ is the edge of $P_a$ incident on $q$ that do not belong to $P_y$, then $e$ belongs to $P_b$ or to $P_x$ or to $P_z$.

Now, assume there exists a vertex $v$ adjacent to both $a$ and $y$, but not to $b$. Then, the clique $\{a, y, v\}$ has to be a claw-clique. Let $q$ be the center of the claw, notice that $q$ has to be an end vertex of the interval $P_a \cap P_y$. Since $v$ is not adjacent to $b$, it follows from the observation at the end of the paragraph above, that $v$ has to be adjacent to $x$ or to $z$.

■

**Lemma 18.** *The graph $F_6$ on Figure 7(f) is not $B_1$-EPG.*

***Proof.*** Let $K = \{1, 2\}$ and $I = \{3, 4, 5\}$. If there exists a $B_1$-EPG representation of $F_6$, by Lemma 17, because of the existence of the vertices 6, 7 and 8, none of the vertices 3, 4 and 5 may intersect 1 between the remaining two, thus such a representation does not exist. ■

**Lemma 19.** *The graph $F_7$ on Figure 7(g) is not $B_1$-EPG.*

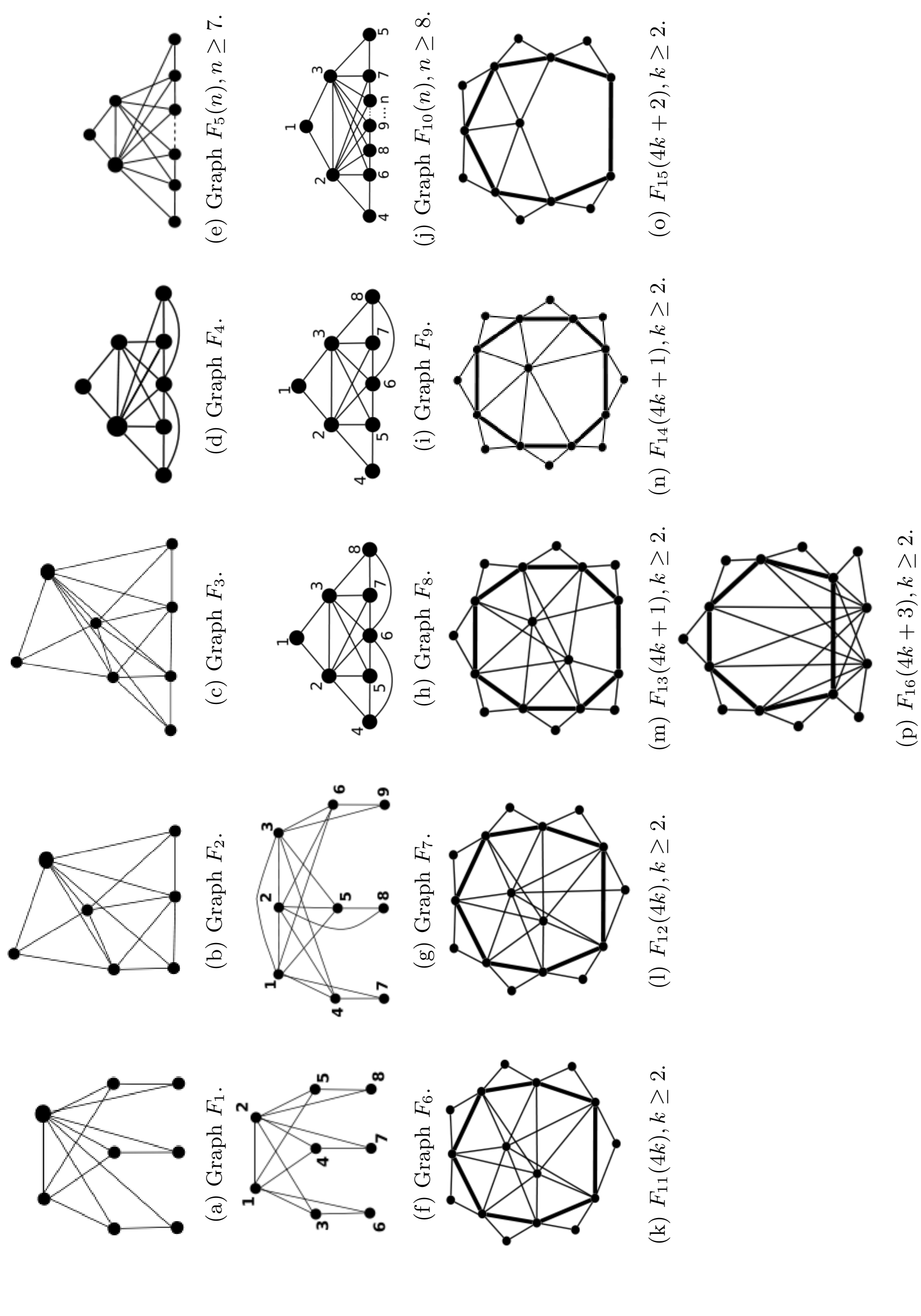
***Proof.*** Let $K = \{1, 2\}$ and $I = \{4, 5, 6\}$. If there exists a $B_1$-EPG representation of $F_7$, by Lemma 17, because of the existence of the vertices 7 and 8, the vertex 6 must intersect vertex 1 between 3 and 4. But considering $K' = \{1, 3\}$, because of the existence of the vertices 5 and 6, vertex 4 must intersect vertex 1 between 5 and 6. This contradiction implies that such a representation does not exist. ∎

**Lemma 20.** *The graphs $F_8$, $F_9$ and $F_{10}(8)$ on Figures 7(h), (i) and (j), respectively, are not $B_1$-EPG.*

***Proof.*** Let $K = \{2, 3\}$ and $I = \{1, 6, 7\}$. If there exists a $B_1$-EPG representation of anyone of those graphs, by Lemma 17, because of the existence of the vertices 4 and 5, the vertex 1 must intersect vertex 2 between 6 and 7. In addition, since $\{2, 6, 8\}$ is a clique, 8 intersects 2 in an edge of $P_6 \cap P_2$ (edge-clique) or in an edge incident to $P_6 \cap P_2$ (claw-clique). Analogously, because of the clique $\{2, 7, 8\}$, 8 intersects 2 in an edge of $P_7 \cap P_2$ (edge-clique) or in an edge incident to $P_7 \cap P_2$ (claw-clique). In any case, it implies that 8 intersects 2 on two different edges, each one in a different side of $P_2 \cap P_1$, thus, by Observation 16, $P_8$ contains the interval $P_2 \cap P_1$, in contradiction with the fact that 1 and 8 are not adjacent. ∎

**Lemma 21.** *The graphs $F_{10}(n)$ for $n \geq 8$ on Figure 7(j) are not $B_1$-EPG.*

***Proof.*** The case $n = 8$ was considered in the previous Lemma 20, so assume $n \geq 9$. Let $K = \{2, 3\}$ and $I = \{1, 6, 7\}$. If there exists a $B_1$-EPG representation of anyone of those graphs, by Lemma 17, because of the existence of the vertices 4 and 5, the vertex 1 must intersect vertex 2 between 6 and 7. In addition, since $\{2, 6, 8\}$ is a clique, 8 intersects 2 in an edge of $P_6 \cap P_2$ (edge-clique) or in an edge incident to $P_6 \cap P_2$ (claw-clique). Analogously, because of the clique $\{2, 7, n\}$, $n$ intersects 2 in an edge of $P_7 \cap P_2$ (edge-clique) or in an edge incident to $P_7 \cap P_2$ (claw-clique). In any case, it implies that 8 and $n$ intersect 2 on two different edges, each one in a different side of $P_2 \cap P_1$. Therefore, there exist two consecutive vertices of the path $8, 9, \ldots, n$, say the vertices $j$ and $j + 1$, such that each one intersects $P_2$ on a different side of $P_2 \cap P_1$. Thus, by Observation 15, $P_j$ or $P_{j+1}$ must contain the interval $P_2 \cap P_1$, in contradiction with the fact that neither $j$ nor $j + 1$ is adjacent to 1. ∎




(a) Graph $F_1$. (b) Graph $F_2$. (c) Graph $F_3$. (d) Graph $F_4$. (e) Graph $F_5(n), n \geq 7$.

(f) Graph $F_6$. (g) Graph $F_7$. (h) Graph $F_8$. (i) Graph $F_9$. (j) Graph $F_{10}(n), n \geq 8$.

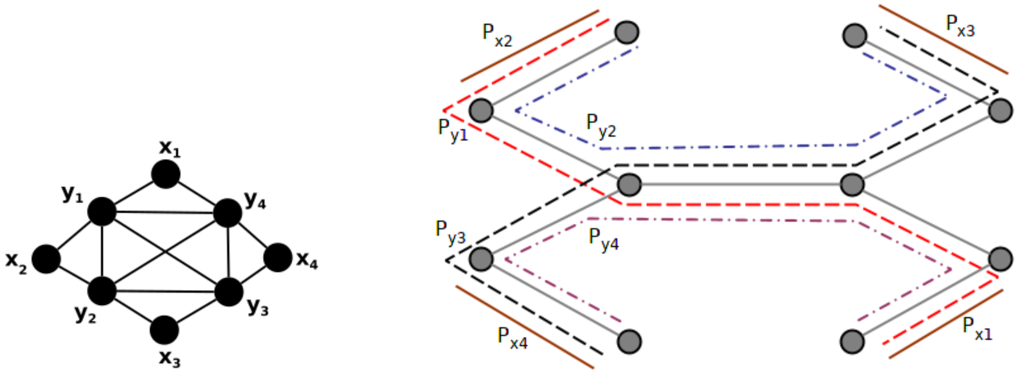
(k) $F_{11}(4k), k \geq 2$. (l) $F_{12}(4k), k \geq 2$. (m) $F_{13}(4k+1), k \geq 2$. (n) $F_{14}(4k+1), k \geq 2$. (o) $F_{15}(4k+2), k \geq 2$.

(p) $F_{16}(4k+3), k \geq 2$.

Figure 7. The 16 Chordal induced subgraphs forbidden to VPT (the vertices in the cycle marked by bold edges form a clique).

We have proved that every minimal forbidden induced subgraph for VPT is also a forbidden induced subgraph for Chordal $B_1$-EPG. Moreover, there are graphs in VPT that do not belong to $B_1$-EPG, for instance the graph 4-sun $S_4$ is not in $B_1$-EPG, see [12], but it has a VPT representation, see Figures 8(a) and 8(b). Thus, VPT graphs properly contain Chordal $B_1$-EPG graphs. This ends the proof of Theorem 14.

**Corollary 22.** *Each one of the graphs depicted on Figure 7 is a forbidden induced subgraph for the class $B_1$-EPG.*




(a) Graph $S_4$. (b) A VPT and EPT representation of $S_4$.

Figure 8. Graph $S_4$ and one of its possible VPT and EPT representations.

**Theorem 23.** *Chordal $B_1$-EPG $\subsetneq$ EPT.*

***Proof.*** Let $G$ be a Chordal $B_1$-EPG graph. By the previous Theorem 14, $G$ is VPT. And, by Lemma 7, $\chi(B(G/C)) \leq 3$ for every maximal clique $C$ of $G$. In [1] (see Theorem 10), it was proved that if the chromatic number of the branch graph of a VPT graph is at most $h$ for every maximal clique, then the graph admits a VPT representation on a host tree with maximum degree $h$. Therefore, $G$ admits a VPT representation on a host tree with maximum degree 3. Finally, in [10] (see Theorem 2), it was proved that any VPT graph that admits a representation on a host tree with maximum degree 3 is also an EPT graph. Consequently, $G$ is EPT.

The same graph $S_4$ used in the proof of the previous theorem (see Figure 8(b)) shows that there are EPT graphs that are not $B_1$-EPG. ■

## 5. Conclusion and Open Questions

In this paper, we have considered three different path-intersection graph classes: $B_1$-EPG, VPT and EPT graphs. We showed that $\{S_3, S_{3'}, S_{3''}, C_4\}$-free graphs and others non-trivial subclasses of $B_1$-EPG graphs are Helly-$B_1$-EPG, namely by instance Bipartite, Block, Cactus and Line of Bipartite graphs.

We presented an infinite family of forbidden induced subgraphs for the class $B_1$-EPG and in particular we proved that Chordal $B_1$-EPG $\subset$ VPT $\cap$ EPT.

In [3], Asinowski and Ries described the Split graphs that are $B_1$-EPG graphs in case the stable set or the central size have size three. The graphs $F_2, F_{11}, F_{13}, F_{14}$ and $F_{15}$, given in Figure 7 are Split, we have used a different approach to prove that they are not $B_1$-EPG graphs. So one question is pertinent: Can we characterize Split graphs in general based on the results of this paper?

Finally, another interesting research would be to explore families of Helly-EPG graphs more deeply. We would like to understand the behavior of other graph classes inside $B_1$-EPG graph class, i.e. if given an input graph $G$ that is an instance of (for example) Weakly Chordal $B_1$-EPG. What is the relationship of $G$ with the EPT/VPT graph class? What happens when we demand that the representations be Helly-$B_1$-EPG? Does recognizing problem remains hard for each one of these classes?

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The present work was done while the third author was a doctoral research fellow at National University of La Plata - UNLP, Math Department. The support of this institution is gratefully acknowledged.

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# Chapter 6

# Concluding Remarks

*If I have seen further, it is by standing upon the shoulders of giants.*

---

Sir Isaac Newton

In Chapter 3, we show that every graph admits a Helly-EPG representation, in particular, is possible to modify the demonstration to prove that every graph admits a monotonic Helly-EPG representation, and $\frac{\mu}{2n} - 1 \leq b_H(G) \leq \mu - 1$. Besides, we relate Helly-$B_1$-EPG graphs with L-shaped graphs, a natural family of subclasses of $B_1$-EPG. Also, we prove that recognizing (Helly-)$B_k$-EPG graphs is in $\mathcal{NP}$, for every fixed $k$. Finally, we show that recognizing Helly-$B_1$-EPG graphs is $NP$-complete, and it remains $NP$-complete even when restricted to 2-apex and 3-degenerate graphs. In addition, at the end of the chapter, we proof that Helly-$B_k$-EPG $\subsetneq$ $B_k$-EPG for each $k > 0$.

In this way, we suggest asking about the complexity of recognizing Helly-$B_k$-EPG graphs for each $k > 1$. Also, it seems interesting to present characterizations for Helly-$B_k$-EPG representations similar to Lemma 6 (especially for $k = 2$, paper of Chapter 3) as well as considering the $h$-Helly-$B_k$ EPG graphs. Regarding L-shaped graphs, it also seems interesting to analyze the classes Helly-[$\llcorner, \ulcorner$] and Helly-[$\llcorner, \ulcorner, \urcorner$] (recall Thereom 14, also paper of Chapter 3).

In Chapter 4, we have determined the Helly number and strong Helly number of $B_k$-EPG graphs and $B_k$-VPG graphs, for $k \geq 0$.

Table 6.1 summarizes the results obtained.

We leave two questions to be investigated concerning the presented results.

1. Given a *specific* EPG or VPG graph, the question is to formulate an algorithm to determine its Helly and strong Helly numbers. See [29], for instance, for such algorithms, applied to general graphs.

2. The values of the Helly and strong Helly numbers, which were determined in

Table 6.1: Helly and Strong Helly Numbers for $B_k$-EPG and $B_k$-VPG Graphs

| $k$ | $B_k$-EPG | $B_k$-VPG |
|---|---|---|
| 0 | 2 | 2 |
| 1 | 3 | 4 |
| 2 | 4 | 6 |
| 3 | 8 | 12 |
| $\geq 4$ | unbounded | unbounded |

the chapter, coincided in all cases. Clearly, in general, this is not the case. We leave as an open question, to find the conditions for such equality to occur.

In Chapter 5, we have considered graphs of the intersection of paths, in particular, Chordal $B_1$-EPG, VPT, and EPT graphs. We show that graphs $\{S_3, S_{3'}, S_{3''}, C_4\}$-free and others non-trivial subclasses of $B_1$-EPG graphs have the Helly property, namely for instance Bipartite, Block, Cactus and Line of Bipartite graphs.

In addition, combining the results of [2, 8, 46] and some other proofs presented in the chapter, we demonstrate by Theorems 14 and 23 (paper of Chapter 5) that Chordal $B_1$-EPG graphs are simultaneously contained in the classes of VPT and EPT graphs.

Asinowski and Ries present in [65] some characterization for special cases of Split $B_1$-EPG graphs, when the stable set has size three or when the clique has size three. Observe that the graphs $F_2, F_{11}, F_{13}, F_{14}, F_{15}$, given in Figure 7 (paper of Chapter 5), are Split but we used another strategy to prove that they are not $B_1$-EPG graphs. So one question is pertinent: Can we characterize Split graphs in general based on results of this chapter?

Another interesting research would be to explore families of Helly-EPG graphs more deeply. We would like to understand the behavior of other graph classes inside $B_1$-EPG graph class, i.e. if given an input graph $G$ that is an instance of (for example) Weakly Chordal $B_1$-EPG. What is the relationship of $G$ with the EPT/VPT graph class? What happens when we demand that the representations be Helly-$B_1$ EPG? Do the recognition problems remain hard for each one of these classes?

In the course of this research, in particular, we studied edge-intersection graphs of paths in a grid such that the paths had at most one bend and the representation has the Helly property for the edges of the paths. The problem of recognizing whether a graph has a $B_k$-EPG representation is an open problem for $k \geq 3$, i.e. given a graph $G$, which is the smallest $k$ such that $G$ has a $B_k$-EPG representation? Also, the problem of recognizing whether a graph has a Helly-$B_k$-EPG representation remains an open problem for $k \geq 2$. The evidence observed in the EPG graph literature and the results obtained in this work makes us conjecture that the problem of recognizing both $B_k$-EPG and Helly-$B_k$-EPG are both $NP$-complete problems, but this demonstration is unknown.

The study of the parameters Helly number and strong Helly number for edge-intersection graphs on a grid was mentioned only in [46, 47], which studied only the parameter strong Helly number. It is easy to see that the questions related of this parameters arise naturally when studying the property of the intersecting sets having the property of being $k$-Helly, thus, another research proposed as the objective of this work was the study of upper and lower bounds for the parameters Helly number and strong Helly number, both for specific classes of EPG and Helly-EPG graphs and also for VPG and Helly-VPG graphs.

In the work of COHEN *et al.* [26], mentioned in Chapter 3, the Cographs that are $B_1$-EPG are characterized by a minimal family of forbidden subgraphs. Moreover, when considered in the context of this work, we can ask: concerning to characterization, what are the Cographs Helly-$B_1$ EPG? Is its recognition also polynomial and can it be done using its co-tree? Is there a difference among these $B_1$-EPG and Helly-$B_1$ EPG families? In addition to the known results for Cographs, we propose potential research topics as problems of recognition or hardness proof for specific classes of graphs $B_1$-EPG and Helly-$B_1$ EPG.

Last but not least, the author of this thesis (Tanilson) conducted research as a sandwich doctorate at the National University of La Plata - UNLP, Argentina, for 1 year (March/2019 until March/2020). The welcome, insertion in the research and workgroup developed during this period must be gratefully acknowledged. Conducting this research at UNLP brought benefits to this doctoral thesis and to the maturity its author as a researcher, since from this period two articles emerged submitted to the SBPO and to DMGT. To continue these works, we hope to explore the Helly-EPG subfamilies.

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